Second-Order Optimization Algorithms

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Chapters 10, 5.4-7, 6.6, Chapter 15

Newton's Method: a Second-Order Method

For multi-variables, Newton's method for minimizing $f(\mathbf{x})$ is to minimize the second-order Taylor expansion function at point $\mathbf{x}^k$:

$$\mathbf{x}^{k+1} = \mathbf{x}^k - (\nabla^2 f(\mathbf{x}^k))^{-1}\nabla f(\mathbf{x}^k).$$

We now introduce the second-order β-Lipschitz condition: for any point $\mathbf{x}$ and direction vector $\mathbf{d}$

$$\|\nabla f(\mathbf{x}+\mathbf{d}) - \nabla f(\mathbf{x}) - \nabla^2 f(\mathbf{x})\mathbf{d}\| \leq \beta\|\mathbf{d}\|^2,$$

which implies

$$f(\mathbf{x}+\mathbf{d}) - f(\mathbf{x}) \leq \nabla f(\mathbf{x})^T\mathbf{d} + \frac{1}{2}\mathbf{d}^T\nabla^2 f(\mathbf{x})\mathbf{d} + \frac{\beta}{3}\|\mathbf{d}\|^3.$$

In the following, for notation simplicity, we use $\mathbf{g}(\mathbf{x}) = \nabla f(\mathbf{x})$ and $\nabla \mathbf{g}(\mathbf{x}) = \nabla^2 f(\mathbf{x})$. Thus,

$$\mathbf{x}^{k+1} = \mathbf{x}^k - (\nabla \mathbf{g}(\mathbf{x}^k))^{-1}\nabla f(\mathbf{x}^k), \text{ or } \nabla \mathbf{g}(\mathbf{x}^k)(\mathbf{x}^{k+1} - \mathbf{x}^k) = -\mathbf{g}(\mathbf{x}^k).$$

Indeed, Newton's method was initially developed for solving a system of nonlinear equations in the form $\mathbf{g}(\mathbf{x}) = \mathbf{0}$.

Local Convergence Theorem of Newton's Method

Theorem 1 *Let $f(\mathbf{x})$ be β-Lipschitz and the smallest absolute eigenvalue of its Hessian uniformly bounded below by $\lambda_{min} > 0$. Then, provided that $\|\mathbf{x}^0 - \mathbf{x}^*\|$ is sufficiently small, the sequence generated by Newton's method converges quadratically to $\mathbf{x}^*$ that is a KKT solution with $\mathbf{g}(\mathbf{x}^*) = \mathbf{0}$.*

$$\begin{aligned}
\|\mathbf{x}^{k+1} - \mathbf{x}^*\| &= \|\mathbf{x}^k - \mathbf{x}^* - \nabla \mathbf{g}(\mathbf{x}^k)^{-1}\mathbf{g}(\mathbf{x}^k)\| \\
&= \|\nabla \mathbf{g}(\mathbf{x}^k)^{-1}\left(\mathbf{g}(\mathbf{x}^k) - \nabla \mathbf{g}(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{x}^*)\right)\| \\
&= \|\nabla \mathbf{g}(\mathbf{x}^k)^{-1}\left(\mathbf{g}(\mathbf{x}^k) - \mathbf{g}(\mathbf{x}^*) - \nabla \mathbf{g}(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{x}^*)\right)\| \\
&\le \|\nabla \mathbf{g}(\mathbf{x}^k)^{-1}\|\|\mathbf{g}(\mathbf{x}^k) - \mathbf{g}(\mathbf{x}^*) - \nabla \mathbf{g}(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{x}^*)\| \\
&\le \|\nabla \mathbf{g}(\mathbf{x}^k)^{-1}\|\beta\|\mathbf{x}^k - \mathbf{x}^*\|^2 \le \frac{\beta}{\lambda_{min}}\|\mathbf{x}^k - \mathbf{x}^*\|^2.
\end{aligned} \tag{1}$$

Thus, when $\frac{\beta}{\lambda_{min}}\|\mathbf{x}^0 - \mathbf{x}^*\| < 1$, the quadratic convergence takes place:

$$\frac{\beta}{\lambda_{min}}\|\mathbf{x}^{k+1} - \mathbf{x}^*\| \le \left(\frac{\beta}{\lambda_{min}}\|\mathbf{x}^k - \mathbf{x}^*\|\right)^2.$$

Such a starting solution $\mathbf{x}^0$ is called an approximate root of $\mathbf{g}(\mathbf{x})$.

Case I: Newton's Method for Computing "Analytic Center"

Consider the optimization problem

$$
\begin{array}{ll}
\text{min} & -\sum_j \ln x_j \\
\text{s.t.} & A\mathbf{x} - \mathbf{b} = \mathbf{0} \in R^m, \\
& \mathbf{x} \geq \mathbf{0}.
\end{array}
$$

Note this is a (strict) convex optimization problem. Suppose the feasible region has an interior and it is bounded, then the (unique) minimizer is called the analytic center of the feasible region, and it, together with multipliers $\mathbf{y}$, $\mathbf{s}$, satisfy the following optimality conditions:

$$
\begin{array}{rcl}
x_j s_j & = & 1, \; j = 1, ..., n, \\
A\mathbf{x} & = & \mathbf{b}, \\
A^T\mathbf{y} + \mathbf{s} & = & \mathbf{0}, \\
(\mathbf{x}, \mathbf{s}) & \geq & \mathbf{0}.
\end{array}
$$

Newton Direction of Linear System

Let $(\mathbf{x} > \mathbf{0}, \mathbf{y}, \mathbf{s} > \mathbf{0})$ be an initial point. Then, the Newton direction would be solution of:

$$\begin{pmatrix} S & \mathbf{0} & X \\ A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A^T & I \end{pmatrix} \begin{pmatrix} \mathbf{d}_x \\ \mathbf{d}_y \\ \mathbf{d}_s \end{pmatrix} = \begin{pmatrix} \mathbf{e} - X\mathbf{s} \\ \mathbf{b} - A\mathbf{x} \\ -A^T\mathbf{y} - \mathbf{s} \end{pmatrix}.$$

Let us assume that the initial point are feasible. Then

$$\begin{aligned} S\mathbf{d}_x + X\mathbf{d}_s &= \mathbf{e} - X\mathbf{s}, \\ A\mathbf{d}_x &= \mathbf{0}, \\ A^T\mathbf{d}_y + \mathbf{d}_s &= \mathbf{0}. \end{aligned} \tag{2}$$

Multiplying AS^{-1} to the top equation and noting $A\mathbf{d}_x = \mathbf{0}$, we have $AXS^{-1}\mathbf{d}_s = AS^{-1}(\mathbf{e} - X\mathbf{s})$, which together with the third equation give

$$\mathbf{d}_y = -(AXS^{-1}A^T)^{-1}AS^{-1}(\mathbf{e} - X\mathbf{s}),$$

$$\mathbf{d}_s = -A^T\mathbf{d}_y, \quad \text{and} \quad \mathbf{d}_x = S^{-1}(\mathbf{e} - X\mathbf{s} - X\mathbf{d}_s).$$

The new Newton iterate would be $\mathbf{x}^+ = \mathbf{x} + \mathbf{d}_x, \quad \mathbf{y}^+ = \mathbf{y} + \mathbf{d}_y, \quad \mathbf{s}^+ = \mathbf{s} + \mathbf{d}_s.$

Quadratic Convergnce from Approximate Centers

The error residual of the first equation would be:

$$\eta(\mathbf{x}, \mathbf{s}) := \|X\mathbf{s} - \mathbf{e}\|. \tag{3}$$

We now prove the following theorem

Theorem 2 *If the starting point of the Newton procedure satisfies* $\eta(\mathbf{x}, \mathbf{s}) < 2/3$, *then*

$$\mathbf{x}^+ > \mathbf{0}, \quad A\mathbf{x}^+ = \mathbf{b}, \quad \mathbf{s}^+ = \mathbf{c}^T - A^T\mathbf{y}^+ > \mathbf{0}$$

and

$$\eta(\mathbf{x}^+, \mathbf{s}^+) \le \frac{\sqrt{2}\eta(\mathbf{x}, \mathbf{s})^2}{4(1 - \eta(\mathbf{x}, \mathbf{s}))} .$$

(ACpd_newton.m of Chapter 5)

Case II: Spherical Trust-Region Method for Minimizing Lipschitz $f(\mathbf{x})$

Recall the second-order β-Lipschitz condition: for any two points $\mathbf{x}$ and $\mathbf{d}$

$$\|\mathbf{g}(\mathbf{x}+\mathbf{d})-\mathbf{g}(\mathbf{x})-\nabla\mathbf{g}(\mathbf{x})\mathbf{d}\| \le \beta\|\mathbf{d}\|^2,$$

where $\mathbf{g}(\mathbf{x})=\nabla f(\mathbf{x})$ and $\nabla\mathbf{g}(\mathbf{x})=\nabla^2 f(\mathbf{x})$. It implies

$$f(\mathbf{x}+\mathbf{d})-f(\mathbf{x}) \le \nabla f(\mathbf{x})^T\mathbf{d}+\frac{1}{2}\mathbf{d}^T\nabla^2 f(\mathbf{x})\mathbf{d}+\frac{\beta}{3}\|\mathbf{d}\|^3.$$

$$\begin{array}{rl}
& f(\mathbf{x}+\mathbf{d})-f(\mathbf{x})-\nabla f(\mathbf{x})^T\mathbf{d}-\frac{1}{2}\mathbf{d}^T\nabla^2 f(\mathbf{x})\mathbf{d} \\
= & \int_0^1 \mathbf{d}^T(\nabla f(\mathbf{x}+t\mathbf{d})-\nabla f(\mathbf{x}))\mathrm{dt}-\frac{1}{2}\mathbf{d}^{\mathrm{T}}\nabla^2\mathrm{f}(\mathbf{x})\mathbf{d} \\
= & \int_0^1 \mathbf{d}^T\left(\nabla f(\mathbf{x}+t\mathbf{d})-\nabla f(\mathbf{x})-\nabla^2 f(\mathbf{x})(t\mathbf{d})\right)\mathrm{dt} \\
\le & \int_0^1 \|\mathbf{d}\|\|\nabla f(\mathbf{x}+t\mathbf{d})-\nabla f(\mathbf{x})-\nabla^2 f(\mathbf{x})(t\mathbf{d})\|\mathrm{dt} \\
\le & \int_0^1 \|\mathbf{d}\|\beta\|t\mathbf{d}\|^2\mathrm{dt}\text{ (by 2nd-order -Lipschitz condition)} \\
= & \beta\|\mathbf{d}\|^3\int_0^1 t^2\mathrm{dt}=\frac{\beta}{3}\|\mathrm{d}\|^3.
\end{array}$$

The second-order method, at the kth iterate, would let $\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}^k$ where

$$\begin{array}{rll} \mathbf{d}^k = & \text{arg min}_{\mathbf{d}} & (\mathbf{c}^k)^T\mathbf{d} + \frac{1}{2}\mathbf{d}^T Q^k \mathbf{d} + \frac{\beta}{3}\alpha^3 \\ & \text{s.t.} & \|\mathbf{d}\| \le \alpha, \end{array}$$

with $\mathbf{c}^k = \nabla f(\mathbf{x}^k)$ and $Q^k = \nabla^2 f(\mathbf{x}^k)$. One typically fixed α to a "trusted" radius α^k so that it becomes a sphere-constrained problem (the inequality is normally active if the Hessian is non PSD):

$$(Q^k + \lambda^k I)\mathbf{d}^k = -\mathbf{c}^k, \; (Q^k + \lambda^k I) \succeq \mathbf{0}, \; \|\mathbf{d}^k\|_2^2 = (\alpha^k)^2.$$

For fixed α^k, the method is generally called trust-region method.

The Trust-Region can be ellipsoidal such as $\|S\mathbf{d}\| \le \alpha$ where S is a PD diagonal scaling matrix.

Convergence Speed of the Spherical Trust-Region Method

Is there a trusted radius such that the method converging? A simple choice would fix $\alpha^k = \sqrt{\epsilon}/\beta$.

Then from reduction (**??**)

$$f(\mathbf{x}^{k+1}) - f(\mathbf{x}^k) \le -\frac{\lambda^k}{2}\|\mathbf{d}^k\|^2 + \frac{\beta}{3}(\alpha^k)^3 = -\frac{\lambda^k(\alpha^k)^2}{2} + \frac{\beta}{3}(\alpha^k)^3 = -\frac{\lambda^k\epsilon}{2\beta^2} + \frac{\epsilon^{3/2}}{3\beta^2}.$$

Also

$$\begin{aligned}\|\mathbf{g}(\mathbf{x}^{k+1})\| &= \|\mathbf{g}(\mathbf{x}^{k+1}) - (\mathbf{c}^k + Q^k\mathbf{d}^k) + (\mathbf{c}^k + Q^k\mathbf{d}^k)\| \\ &\le \|\mathbf{g}(\mathbf{x}^{k+1}) - (\mathbf{c}^k + Q^k\mathbf{d}^k)\| + \|(\mathbf{c}^k + Q^k\mathbf{d}^k)\| \\ &\le \beta\|\mathbf{d}^k\|^2 + \lambda^k\|\mathbf{d}^k\| = \beta(\alpha^k)^2 + \lambda^k\alpha^k = \frac{\epsilon}{\beta} + \frac{\lambda^k\sqrt{\epsilon}}{\beta}.\end{aligned}$$

Thus, one can stop the algorithm as soon as $\lambda^k \le \sqrt{\epsilon}$ so that the inequality becomes $\|\mathbf{g}(\mathbf{x}^{k+1})\| \le \frac{2\epsilon}{\beta}$ and the function value is decreased at least $-\frac{\epsilon^{1.5}}{6\beta^2}$. Furthermore, $|\lambda_{min}(\nabla\mathbf{g}(\mathbf{x}^k))| \le \lambda^k = \sqrt{\epsilon}$.

Theorem 3 *Let the objective function* $p^* = \inf\ f(\mathbf{x})$ *be finite. Then in* $\frac{O(\beta^2(f(\mathbf{x}^0)-p^*))}{\epsilon^{1.5}}$ *iterations of the trust-region method, the norm of the gradient vector is less than* ϵ *and the Hessian is* $\sqrt{\epsilon}$*-positive semidefinite, where each iteration solves a spherical-constrained quadratic minimization discussed earlier.*

Relation to Quadratic/Cubic Regularization/Proximal-Point Method

One can also interpret the Spherical Trust-Region method as the Quadratic Regularization Method

$$\mathbf{d}^k(\lambda) = \quad \arg\min_{\mathbf{d}} \quad (\mathbf{c}^k)^T\mathbf{d} + \frac{1}{2}\mathbf{d}^T Q^k \mathbf{d} + \frac{\lambda}{2}\|\mathbf{d}\|^2$$

where parameter λ makes $(Q^k + \lambda I) \succeq \mathbf{0}$. Then consider the one-variable function

$$\phi(\lambda) := f(\mathbf{x}^k + \mathbf{d}^k(\lambda))$$

and do one-variable minimization of $\phi(\lambda)$ over λ. Then let λ^k be a minimizer and $\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}^k(\lambda^k)$.

Thus, based on the earlier analysis, we must have at least

$$f(\mathbf{x}^{k+1}) - f(\mathbf{x}^k) \le -\frac{\epsilon^{1.5}}{6\beta^2}$$

for some (local) Lipschitz constant β of the objective function.

Note that the algorithm needs to estimate only the minimum eigenvalue, $\lambda_{min}(Q^k)$, of the Hessian. One heuristic is to let λ^k decreases geometrically and do few possible line-search steps.

Dimension-Reduced Trust-Region Method in the Subspace

Let $\mathbf{g}^k = \nabla f(\mathbf{x}^k)$ and the momentum directon $\mathbf{d}^k = \mathbf{x}^k - \mathbf{x}^{k-1}$. Again we consider iteration

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_1 \mathbf{g}^k + \alpha_2 \mathbf{d}^k = \mathbf{x}^k + \mathbf{d}(\alpha)$$

where the pair of step-sizes $\alpha = (\alpha_1; \alpha_2)$ can be chosen from

$$\min_{\alpha \in S^k} \ \psi^k(\alpha) := \nabla f(\mathbf{x}^k)\mathbf{d}(\alpha) + \frac{1}{2}\mathbf{d}(\alpha)\nabla^2 f(\mathbf{x}^k)\mathbf{d}(\alpha).$$

Here S^k is a trust-region of α for the two-dimensional quadratic prolem that has Hessian and gradient

$$\nabla^2 \psi^k(\alpha) = \begin{pmatrix} (\mathbf{g}^k)^T H^k \mathbf{g}^k & -(\mathbf{d}^k)^T H^k \mathbf{g}^k \\ -(\mathbf{d}^k)^T H^k \mathbf{g}^k & (\mathbf{d}^k)^T H^k \mathbf{d}^k \end{pmatrix}, \nabla \psi^k(\alpha) = \begin{pmatrix} -\|\mathbf{g}^k\|^2 \\ (\mathbf{g}^k)^T \mathbf{d}^k \end{pmatrix}.$$

wehere $H^k = \nabla^2 f(\mathbf{x}^k)$. Again, if the full Hessian is not available, one can approximate $\nabla^2 \psi^k(\alpha)$ using

$$H^k \mathbf{g}^k \sim \nabla(\mathbf{x}^k + \mathbf{g}^k) - \mathbf{g}^k \quad \text{and} \quad H^k \mathbf{d}^k \sim \nabla(\mathbf{x}^k + \mathbf{d}^k) - \mathbf{g}^k \sim -(\mathbf{g}^{k-1} - \mathbf{g}^k);$$

or more accurate difference approximation between two gradients. (DRSOMTrust....m of Chapter 8)

Do the Second-Order Methods Make a Difference

Consider the nonconvex compressed-sensing model

$$\min_{\mathbf{x}} 0.5\|A\mathbf{x} - \mathbf{b}\|^2 + \mu \sum_j |x_j|^p, \text{ s.t. } \mathbf{x} \geq \mathbf{0}$$

where $0 < p \leq 1$.

(TrustL2Lx....m of Chapter 8)

Would Convexity Help SOM?

Before we answer this question, let's summarize a generic form one iteration of the Second Order Method for solving $\nabla f(\mathbf{x}) = \mathbf{g}(\mathbf{x}) = \mathbf{0}$:

$$(\nabla \mathbf{g}(\mathbf{x}^k) + \mu I)(\mathbf{x} - \mathbf{x}^k) = -\gamma \mathbf{g}(\mathbf{x}^k), \quad \text{or}$$

$$\mathbf{g}(\mathbf{x}^k) + \nabla \mathbf{g}(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) + \mu(\mathbf{x} - \mathbf{x}^k) = (1 - \gamma)\mathbf{g}(\mathbf{x}^k).$$

Many interpretations: when

- $\gamma = 1, \mu = 0$: pure Newton;
- γ and μ are sufficiently large: SDM;
- $\gamma = 1$ and μ decreases to 0: Homotopy or path-following method.

A Path-Following Algorithm for Unconstrained Optimization I

For any $\mu > 0$ consider the (unique) optimal solution $\mathbf{x}(\mu)$ for problem

$$\mathbf{x}(\mu) = \text{arg min } f(\mathbf{x}) + \frac{\mu}{2}\|\mathbf{x}\|^2,$$

and they form a path down to $\mathbf{x}(0)$ and satisfy gradient equations with parameter μ:

$$\mathbf{g}(\mathbf{x}) + \mu\mathbf{x} = \mathbf{0}, \quad \text{with} \quad \mu = \mu^k > 0. \tag{4}$$

Let the approximation path error at $\mathbf{x}^k$ with $\mu = \mu^k$ be

$$\|\mathbf{g}(\mathbf{x}^k) + \mu^k\mathbf{x}^k\| \le \frac{1}{2\beta}\mu^k.$$

Then, we like to compute a new iterate $\mathbf{x}^{k+1}$, using Newton's method with $\mathbf{x}^k$ as an initial solution, such that

$$\|\mathbf{g}(\mathbf{x}^{k+1}) + \mu^{k+1}\mathbf{x}^{k+1}\| \le \frac{1}{2\beta}\mu^{k+1}, \quad \text{where } 0 \le \mu^{k+1} < \mu^k.$$

If μ^k can be decreased at a geometric rate, independent of ϵ, and each update uses one Newton step, then this would lead to a linearly convergent algorithm.

Concordant Lipschitz Functions

We analyze the path-following algorithm when f is convex and meet a Concordant Lipschitz condition: for any point $\mathbf{x}$ and a $\beta \geq 1$

$$\|\nabla f(\mathbf{x}+\mathbf{d}) - \nabla f(\mathbf{x}) - \nabla^2 f(\mathbf{x})\mathbf{d}\| \leq \beta \mathbf{d}^T \nabla^2 f(\mathbf{x})\mathbf{d}, \text{ whenever } \|\mathbf{d}\| \leq O(1) < 1 \quad (5)$$

and $\mathbf{x}+\mathbf{d}$ in the function domain. Such condition can be verified using Taylor Expansion Series; basically, the third derivative of the function is bounded by its second derivative.

- All quadratic functions are concordant Lipschitz with $\beta = 0$.
- Convex function e^x is concordant Lipschitz with $\beta = O(1)$ but it is not regular Lipschitz.
- Convex function $-\log(x)$ is neither regular Lipschitz nor concordant Lipschitz.
- Function $f(\mathbf{x}) := \phi(A\mathbf{x} - \mathbf{b})$ is concordant Lipschitz if $\phi(\cdot)$ is regular Lipschitz and strictly convex.

A Path-Following Algorithm for Unconstrained Optimization II

When μ^k is replaced by μ^{k+1}, say $(1-\eta)\mu^k$ for some $\eta \in (0,\ 1]$, we aim to find a solution $\mathbf{x}$ such that

$$\mathbf{g}(\mathbf{x}) + (1-\eta)\mu^k \mathbf{x} = \mathbf{0},$$

we start from $\mathbf{x}^k$ and apply the Newton iteration:

$$\begin{array}{c} \mathbf{g}(\mathbf{x}^k) + \nabla \mathbf{g}(\mathbf{x}^k)\mathbf{d} + (1-\eta)\mu^k(\mathbf{x}^k + \mathbf{d}) = \mathbf{0}, \quad \text{or} \\ \nabla \mathbf{g}(\mathbf{x}^k)\mathbf{d} + (1-\eta)\mu^k \mathbf{d} = -\mathbf{g}(\mathbf{x}^k) - (1-\eta)\mu^k \mathbf{x}^k. \end{array} \tag{6}$$

From the second expression, we have

$$\begin{array}{rcl} \|\nabla \mathbf{g}(\mathbf{x}^k)\mathbf{d} + (1-\eta)\mu^k \mathbf{d}\| & = & \| - \mathbf{g}(\mathbf{x}^k) - (1-\eta)\mu^k \mathbf{x}^k\| \\ & = & \| - \mathbf{g}(\mathbf{x}^k) - \mu^k \mathbf{x}^k + \eta\mu^k \mathbf{x}^k\| \\ & \le & \| - \mathbf{g}(\mathbf{x}^k) - \mu^k \mathbf{x}^k\| + \eta\mu^k\|\mathbf{x}^k\| \\ & \le & \frac{1}{2\beta}\mu^k + \eta\mu^k\|\mathbf{x}^k\|. \end{array} \tag{7}$$

On the other hand

$$\|\nabla \mathbf{g}(\mathbf{x}^k)\mathbf{d} + (1-\eta)\mu^k \mathbf{d}\|^2 = \|\nabla \mathbf{g}(\mathbf{x}^k)\mathbf{d}\|^2 + 2(1-\eta)\mu^k \mathbf{d}^T \nabla \mathbf{g}(\mathbf{x}^k)\mathbf{d} + ((1-\eta)\mu^k)^2\|\mathbf{d}\|^2.$$

From convexity, $\mathbf{d}^T\nabla\mathbf{g}(\mathbf{x}^k)\mathbf{d} \geq 0$, together with (7) we have

$$\begin{array}{rcl} ((1-\eta)\mu^k)^2\|\mathbf{d}\|^2 & \leq & (\frac{1}{2\beta}+\eta\|\mathbf{x}^k\|)^2(\mu^k)^2 \quad \text{and} \\ 2(1-\eta)\mu^k\mathbf{d}^T\|\nabla\mathbf{g}(\mathbf{x}^k)\mathbf{d} & \leq & (\frac{1}{2\beta}+\eta\|\mathbf{x}^k\|)^2(\mu^k)^2. \end{array}$$

The first inequality implies

$$\|\mathbf{d}\|^2 \leq (\frac{1}{2\beta(1-\eta)}+\frac{\eta}{1-\eta}\|\mathbf{x}^k\|)^2.$$

Let the new iterate be $\mathbf{x}^+ = \mathbf{x}^k + \mathbf{d}$. The second inequality implies

$$\begin{array}{rl} & \|\mathbf{g}(\mathbf{x}^+)+(1-\eta)\mu^k\mathbf{x}^+\| \\ = & \|\mathbf{g}(\mathbf{x}^+)-(\mathbf{g}(\mathbf{x}^k)+\nabla\mathbf{g}(\mathbf{x}^k)\mathbf{d})+(\mathbf{g}(\mathbf{x}^k)+\nabla\mathbf{g}(\mathbf{x}^k)\mathbf{d})+(1-\eta)\mu^k(\mathbf{x}^k+\mathbf{d})\| \\ = & \|\mathbf{g}(\mathbf{x}^+)-\mathbf{g}(\mathbf{x}^k)+\nabla\mathbf{g}(\mathbf{x}^k)\mathbf{d}\| \\ \leq & \beta\mathbf{d}^T\nabla\mathbf{g}(\mathbf{x}^k)\mathbf{d} \leq \frac{\beta}{2(1-\eta)}(\frac{1}{2\beta}+\eta\|\mathbf{x}^k\|)^2\mu^k. \end{array}$$

We now just need to choose $\eta \in (0,\ 1)$ such that

$$\begin{array}{rcl} (\frac{1}{2\beta(1-\eta)}+\frac{\eta}{1-\eta}\|\mathbf{x}^k\|)^2 & \leq & 1 \quad \text{and} \\ \frac{\beta\mu^k}{2(1-\eta)}(\frac{1}{2\beta}+\eta\|\mathbf{x}^k\|)^2 & \leq & \frac{1}{2\beta}(1-\eta)\mu^k = \frac{1}{2\beta}\mu^{k+1}. \end{array}$$

For example, given $\beta \geq 1$,

$$\eta = \frac{1}{2\beta(1 + \|\mathbf{x}^k\|)}$$

would suffice.

This would give a linear convergence since $\|\mathbf{x}^k\|$ is typically bounded following the path to the optimality, while the convergence in non-convex case is only arithmetic.

More question related to the path-following algorithm:

- For convex case, since $\mathbf{x}(\mu)$ is the unique minimizer of

  $$\text{min} \;\; f(\mathbf{x}) + \frac{\mu}{2}\|\mathbf{x}\|^2,$$

  what is the limit of $\mathbf{x}(\mu)$ as $\mu \to 0^+$?

- More practical strategy to decrease μ?

- Apply first-order or 1.5-order algorithms for solving each step of the path-following, since it is to minimize a strictly convex quadratic function?

- What happen when f is bounded from below but not convex, and just meet the standard Lipschitz condition? The key is analyzing $\mathbf{x}(\mu)$, which may form multiple paths. Then can we still follow the path?

  (QPpath.m of Chapter 8)

Interior-Point Methods (IPM) for Linear Programming

Optimality Conditions: (1) Primal Feasibility, (2) Dual Feasibility, (3) Zero-Duality Gap/Prima-Dual Complementarity.

Recall that the (primal) Simplex Algorithm maintains the primal feasibility and complementarity while working toward dual feasibility. (The Dual Simplex Algorithm maintains dual feasibility and complementarity while working toward primal feasibility.)

In contrast, interior-point methods will move in the interior of the feasible region, hoping to by-pass many corner points on the boundary of the region. The primal-dual interior-point method maintains both primal and dual feasibility while working toward complementarity.

The key for the simplex method is to make computer see corner points; and the key for interior-point methods is to stay in the interior of the feasible region.

Interior Feasible Points for LP and LD

$$(LP)\ \text{min}\ \mathbf{c}^T\mathbf{x}\ \text{s.t.}\ A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \quad <=> \quad (LD)\ \text{max}\ \mathbf{b}^T\mathbf{y}\ \text{s.t.} A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \geq \mathbf{0}.$$

$$\text{int}\,\mathcal{F}_p = \{\mathbf{x} :\ A\mathbf{x} = \mathbf{b},\ \mathbf{x} > \mathbf{0}\} \neq \emptyset$$

$$\text{int}\,\mathcal{F}_d = \{(\mathbf{y}, \mathbf{s}) :\ \mathbf{s} = \mathbf{c} - A^T\mathbf{y} > \mathbf{0}\} \neq \emptyset.$$

Let z^* denote the optimal value and

$$\mathcal{F} = \mathcal{F}_p \times \mathcal{F}_d.$$

We are interested in finding an ϵ-approximate solution for the LP problem:

$$\mathbf{x}^T\mathbf{s} = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} \leq \epsilon.$$

For simplicity, we assume that an interior-point pair $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{s}^0)$ is known, and we will use it as our initial point pair.

Barrier Functions and Analytic Center Again

Consider the barrier function optimization problems:

$$
\begin{array}{lll} (PB) & \text{minimize} & -\sum_{j=1}^{n} \log x_j \\ & \text{s.t.} & \mathbf{x} \in \text{int}\, \mathcal{F}_p \end{array}
\quad \text{and} \quad
\begin{array}{lll} (DB) & \text{maximize} & \sum_{j=1}^{n} \log s_j \\ & \text{s.t.} & (\mathbf{y}, \mathbf{s}) \in \text{int}\, \mathcal{F}_d \end{array}
$$

The maximizer $\mathbf{x}$ (or $(\mathbf{y}, \mathbf{s})$) of (PB) (or (BD)) is called the analytic center of bounded polyhedron $\mathcal{F}_p$ (or $\mathcal{F}_d$). Applying the KKT conditions and using $X = \text{diag}(\mathbf{x})$, we have

$$
-X^{-1}\mathbf{e} - A^T\mathbf{y} = \mathbf{0} \quad \text{or} \quad -\mathbf{e} - XA^T\mathbf{y} = \mathbf{0},\ A\mathbf{x} = \mathbf{b},\ \mathbf{x} > \mathbf{0}.
$$

After introducing auxiliary vector $\mathbf{s} = X^{-1}\mathbf{e}$, the conditions become

$$
\begin{array}{rcl} X\mathbf{s} & = & \mathbf{e} \\ A\mathbf{x} & = & \mathbf{b} \\ -A^T\mathbf{y} - \mathbf{s} & = & \mathbf{0} \\ \mathbf{x} & > & \mathbf{0}. \end{array}
\qquad
\left(\text{or} \quad \begin{array}{rcl} S\mathbf{x} & = & \mathbf{e} \\ A\mathbf{x} & = & \mathbf{0} \\ -A^T\mathbf{y} - \mathbf{s} & = & -\mathbf{c} \\ \mathbf{s} & > & \mathbf{0}. \end{array} \right)
$$

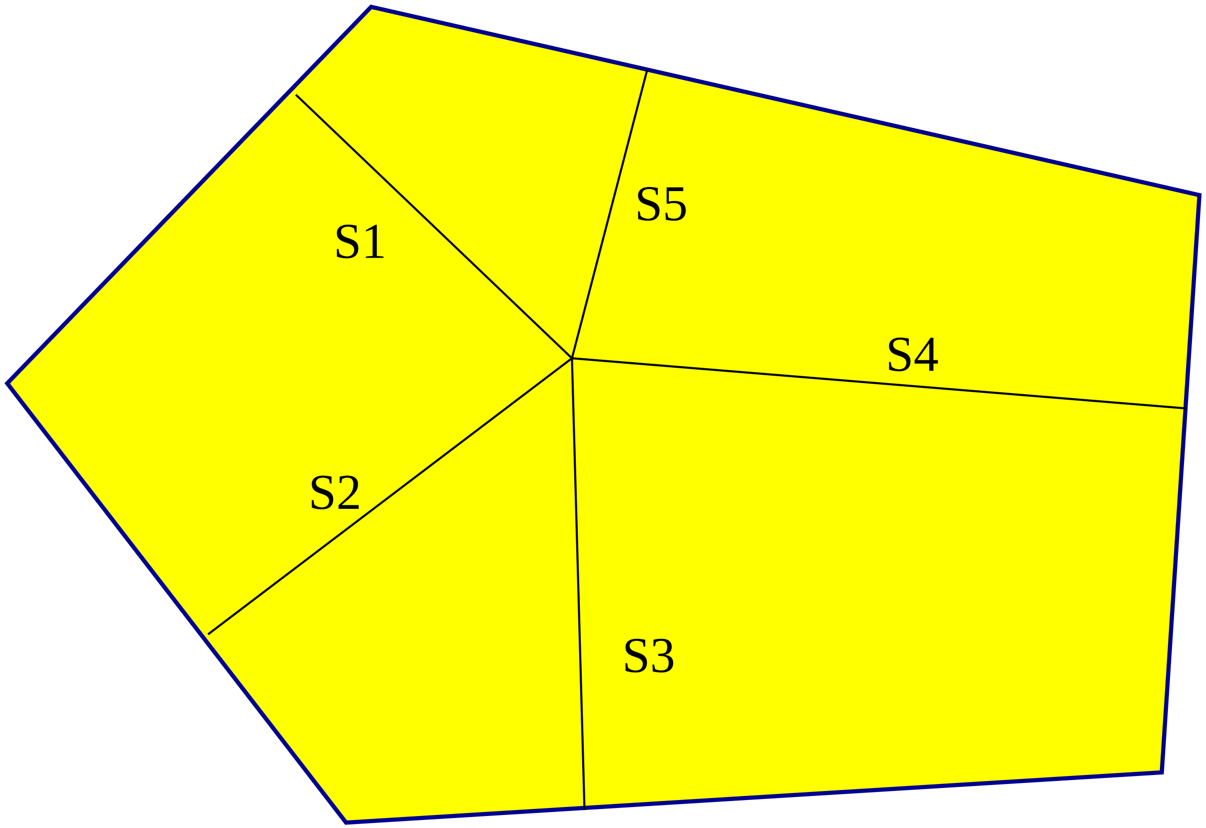

Figure 1: The dual analytic center maximizes the product of slacks.

Examples

$$\mathcal{F}_p = \{\mathbf{x} : \sum_j \mathbf{x}_j = 1, \ \mathbf{x} \geq \mathbf{0}\}.$$

The analytic center of $\mathcal{F}_p$ would be

$$\mathbf{x}^c = (\frac{1}{n}; \ \ldots; \ \frac{1}{n}), \ y = -n, \ \mathbf{s} = (n; \ \ldots; \ n).$$

$$\mathcal{F}_d = \{\mathbf{y} : \mathbf{0} \leq \mathbf{y} \leq \mathbf{e}\}.$$

The analytic center of $\mathcal{F}_d$ would be

$$\mathbf{y}^c = \arg\max \sum_i (\log(y_i) + \log(1 - y_i)) = \arg\max \sum_i \log(y_i(1 - y_i))$$

that is

$$\mathbf{y}^c = (\frac{1}{2}; \ \ldots; \ \frac{1}{2}), \ \mathbf{s} = \frac{1}{2}\mathbf{e}, \ \mathbf{x} = 2\mathbf{e}.$$

Why “analytic”: depending on the analytical representation data.

Logarithmic Function and Scaled Concordant Lipschitz

Lemma 1 *Let* $B(\mathbf{x}) = -\sum_{j=1}^{n} \log(x_j)$*. Then, for any point* $\mathbf{x} > \mathbf{0}$ *and direction vector* $\mathbf{d}$ *such that* $\|X^{-1}\mathbf{d}\|_\infty \leq \alpha(< 1)$,

$$-\mathbf{e}^T X^{-1}\mathbf{d} \leq B(\mathbf{x}+\mathbf{d}) - B(\mathbf{x}) \leq -\mathbf{e}^T X^{-1}\mathbf{d} + \frac{\|X^{-1}\mathbf{d}\|^2}{2(1-\alpha)}.$$

The Barrier function property can be generalized to the so-called Second-Order Scaled Concordant Lipschitz Condition: for any $\mathbf{x} > \mathbf{0}$ and $\mathbf{x} + \mathbf{d}$ in the function domain:

$$\|X\left(\nabla f(\mathbf{x}+\mathbf{d}) - \nabla f(\mathbf{x}) - \nabla^2 f(\mathbf{x})\mathbf{d}\right)\| \leq \beta_\alpha \mathbf{d}^T \nabla^2 f(\mathbf{x})\mathbf{d}, \text{ whenever } \|X^{-1}\mathbf{d}\| \leq \alpha(< 1).$$

Such condition can be verified using Taylor Expansion Series; basically, the scaled third derivative of the function is bounded by its (unscaled) second derivative.

- All quadratic functions are scaled concordant Lipschitz with $\beta_\alpha = 0$.
- Convex function $-\log(x)$ is scaled concordant Lipschitz with $\beta_\alpha = \frac{1}{(1-\alpha)}$.
- All power functions $\{x^p : x > 0\}$ with integer p are scaled concordant Lipschitz with $\beta_\alpha = \frac{O(p)}{(1-\alpha)}$.

Affine-Scaling Gradient Projection

To compute the analytic center, we consider the affine-scaling GPM from any feasible $\mathbf{x} > \mathbf{0}$:

$$\begin{array}{ll} \text{minimize} & -\mathbf{e}^T X^{-1}\mathbf{d} \\ \text{s.t.} & A\mathbf{d} = \mathbf{0},\ \|X^{-1}\mathbf{d}\| \le \alpha \end{array} \quad \text{or} \quad \begin{array}{ll} \text{minimize} & -\mathbf{e}^T \mathbf{d}' \\ \text{s.t.} & AX\mathbf{d}' = \mathbf{0},\ \|\mathbf{d}'\| \le \alpha \end{array}$$

which has a close-form solution

$$\mathbf{d}' = \alpha(I - XA^T(AX^2A^T)^{-1}AX)\mathbf{e}/\|(I - XA^T(AX^2A^T)^{-1}AX)\mathbf{e}\|.$$

Note that $\mathbf{d} = X\mathbf{d}'$ so that we let $\mathbf{x}^+ = \mathbf{x} + \mathbf{d}$, which should remain positive:

$$\mathbf{x}^+ = \mathbf{x} + \mathbf{d} = \mathbf{x} + X\mathbf{d}' = X(\mathbf{e} + \mathbf{d}') > \mathbf{0}$$

as long as $\mathbf{x} > \mathbf{0}$ and $\|\mathbf{d}'\| < 1$. Then, from Lemma 1 the Barrier function value would be decreased at least by

$$B(\mathbf{x}^+) - B(\mathbf{x}) \le -\alpha\|(I - XA^T(AX^2A^T)^{-1}AX)\mathbf{e}\| + \frac{\alpha^2}{2(1-\alpha)}.$$

Convergence Speed Analysis

For simplicity, let $\mathbf{y}(\mathbf{x}) = (AX^2A^T)^{-1}AX\mathbf{e}$ and $\mathbf{s}(\mathbf{x}) = A^T\mathbf{y}(\mathbf{s})$ so that

$$(I - XA^T(AX^2A^T)^{-1}AX)\mathbf{e} = \mathbf{e} - X\mathbf{s}(\mathbf{x}).$$

Note that $\mathbf{y}(\mathbf{x})$ minimizes $\min_{\mathbf{y}} \; \|\mathbf{e} - XA^T\mathbf{y}\|^2$. Thus, as long as $\|\mathbf{e} - X\mathbf{s}(\mathbf{x})\| \geq 1$, the Barrier function can be decreased by a universal constant $-\alpha + \frac{\alpha^2}{2(1-\alpha)} = -3/4$ when we set $\alpha = 1/2$. If the quantity $\|\mathbf{e} - X\mathbf{s}(\mathbf{x})\| < 1$, then we simply let $\mathbf{x}^+ = \mathbf{x} + X(\mathbf{e} - X\mathbf{s}(\mathbf{x}))$, in which case we now prove $\|\mathbf{e} - X^+\mathbf{s}(\mathbf{x}^+)\| \leq \|\mathbf{e} - X\mathbf{s}(\mathbf{x})\|^2$ (quadratic convergence)!

$$\begin{aligned}
\|\mathbf{e} - X^+\mathbf{s}(\mathbf{x}^+)\|^2 &\leq \|\mathbf{e} - X^+\mathbf{s}(\mathbf{x})\|^2, \quad (\text{because } \mathbf{y}(\mathbf{x}^+) \text{ minimizes the squares}) \\
&= \|\mathbf{e} - (2X - X^2S(\mathbf{x})\mathbf{s}(\mathbf{x})\|^2 \\
&= \sum_{j=1}^n \left(1 - 2x_js_j(\mathbf{x}) + x_j^2(s_j(\mathbf{x}))^2\right)^2 \\
&= \sum_{j=1}^n (1 - x_js_j(\mathbf{x}))^4 \\
&\leq \left(\sum_{j=1}^n (1 - x_js_j(\mathbf{x}))^2\right)^2 = \|\mathbf{e} - X\mathbf{s}(\mathbf{x})\|^4.
\end{aligned}$$

(ACprimal....m of Chapter 5)

Analytic Volume and Cutting Plane for LP: Geometric Interpretation of IPM

$$AV(\mathcal{F}_d) := \prod_{j=1}^{n} \bar{s}_j = \prod_{j=1}^{n} (c_j - \mathbf{a}_j^T \bar{\mathbf{y}})$$

can be viewed as the analytic volume of polytope $\mathcal{F}_d$ or simply $\mathcal{F}$ in the rest of discussions.

If one inequality in $\mathcal{F}$, say the first one, needs to be translated, change $\mathbf{a}_1^T \mathbf{y} \le c_1$ to $\mathbf{a}_1^T y \le \mathbf{a}_1^T \bar{\mathbf{y}}$; i.e., the first inequality is parallelly moved and it now cuts through $\bar{\mathbf{y}}$ and divides $\mathcal{F}$ into two bodies. Analytically, c_1 is replaced by $\mathbf{a}_1^T \bar{\mathbf{y}}$ and the rest of data are unchanged. Let

$$\mathcal{F}^+ := \{\mathbf{y} : \mathbf{a}_j^T \mathbf{y} \le c_j^+, \; j = 1, ..., n\},$$

where $c_j^+ = c_j$ for $j = 2, ..., n$ and $c_1^+ = \mathbf{a}_1^T \bar{\mathbf{y}}$.

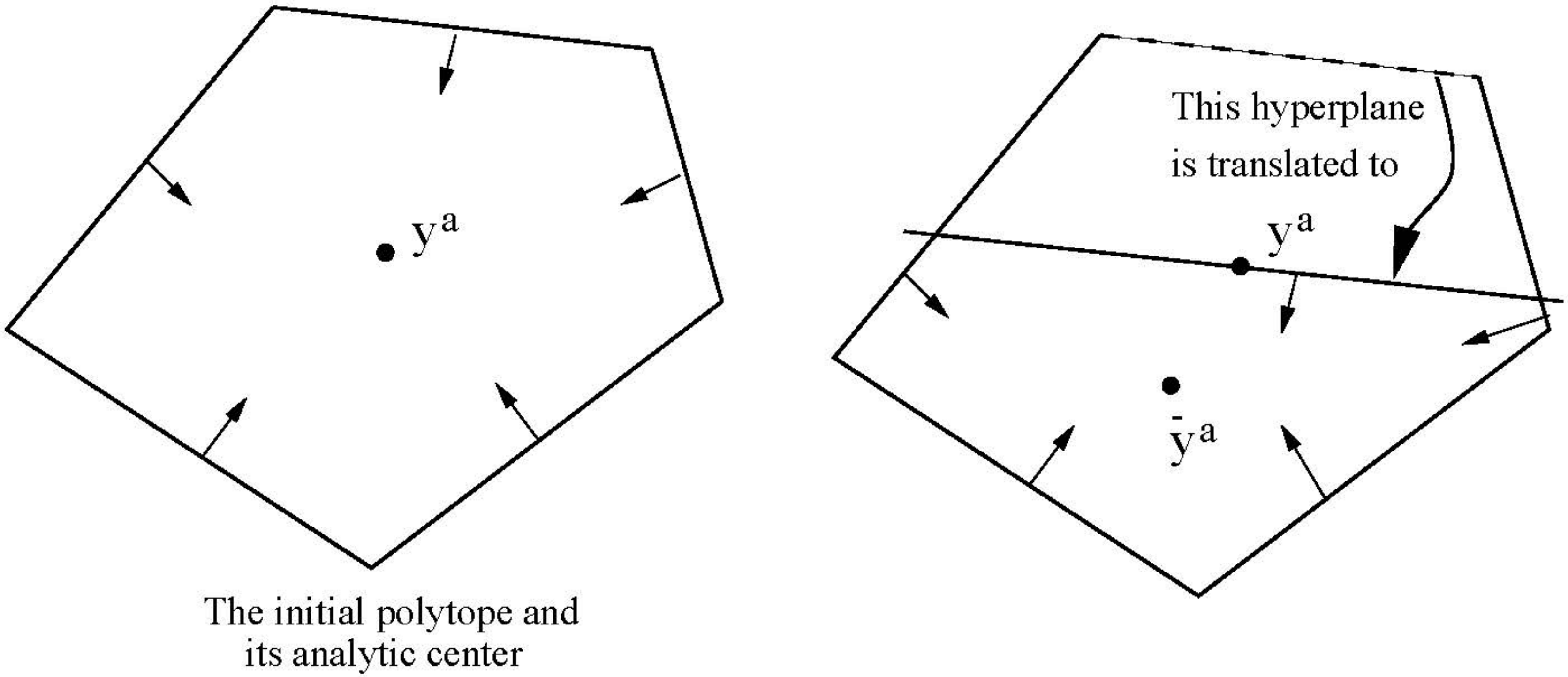

Figure 2: Translation of a hyperplane to the AC.

Analytic Volume Reduction of the New Polytope

Let $\bar{\mathbf{y}}^+$ be the analytic center of $\mathcal{F}^+$. Then, the analytic volume of $\mathcal{F}^+$

$$AV(\mathcal{F}^+) = \prod_{j=1}^{n}(c_j^+ - \mathbf{a}_j^T\bar{\mathbf{y}}^+) = (\mathbf{a}_1^T\bar{\mathbf{y}} - \mathbf{a}_1^T\bar{\mathbf{y}}^+)\prod_{j=2}^{n}(c_j - \mathbf{a}_j^T\bar{\mathbf{y}}^+).$$

We have the following volume reduction theorem:

Theorem 4

$$\frac{AV(\mathcal{F}^+)}{AV(\mathcal{F})} \le \exp(-1).$$

Now suppose we translate $k(< n)$ hyperplanes, say $1, 2, ..., k$, moved to cut the analytic center $\bar{\mathbf{y}}$ of $\mathcal{F}$, that is,

$$\mathcal{F}^+ := \{\mathbf{y} :\ \mathbf{a}_j^T\mathbf{y} \le c_j^+,\ j = 1, ..., n\},$$

where $c_j^+ = c_j$ for $j = k+1, ..., n$ and $c_j^+ = \mathbf{a}_j^T\bar{\mathbf{y}}$ for $j = 1, ..., k$.

Corollary 1

$$\frac{AV(\mathcal{F}^+)}{AV(\mathcal{F})} \le \exp(-k).$$

Barrier Regularization Function for LP: Algebraic Implementation of IPM

Consider the LP pair with the barrier function

$$\begin{array}{lll} (LPB) & \text{minimize} & \mathbf{c}^T\mathbf{x} - \mu\sum_{j=1}^n \log x_j \\ & \text{s.t.} & \mathbf{x} \in \text{int}\,\mathcal{F}_p \end{array} \quad <=> \quad \begin{array}{lll} (LDB) & \text{maximize} & \mathbf{b}^T\mathbf{y} + \mu\sum_{j=1}^n \log s_j \\ & \text{s.t.} & (\mathbf{y}, \mathbf{s}) \in \text{int}\,\mathcal{F}_d, \end{array}$$

and they are primal-dual to each other and share a common set of KKT Optimality Conditions:

$$\begin{array}{rcl} X\mathbf{s} & = & \mu\mathbf{e} \\ A\mathbf{x} & = & \mathbf{b} \\ -A^T\mathbf{y} - \mathbf{s} & = & -\mathbf{c}; \end{array} \qquad (8)$$

where barrier parameter

$$\mu = \frac{\mathbf{x}^T\mathbf{s}}{n} = \frac{\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}}{n},$$

so that it's the average of complementarity or duality gap. As μ varies, the optimizers form the LP central paths in the primal and dual feasible regions, respectively.

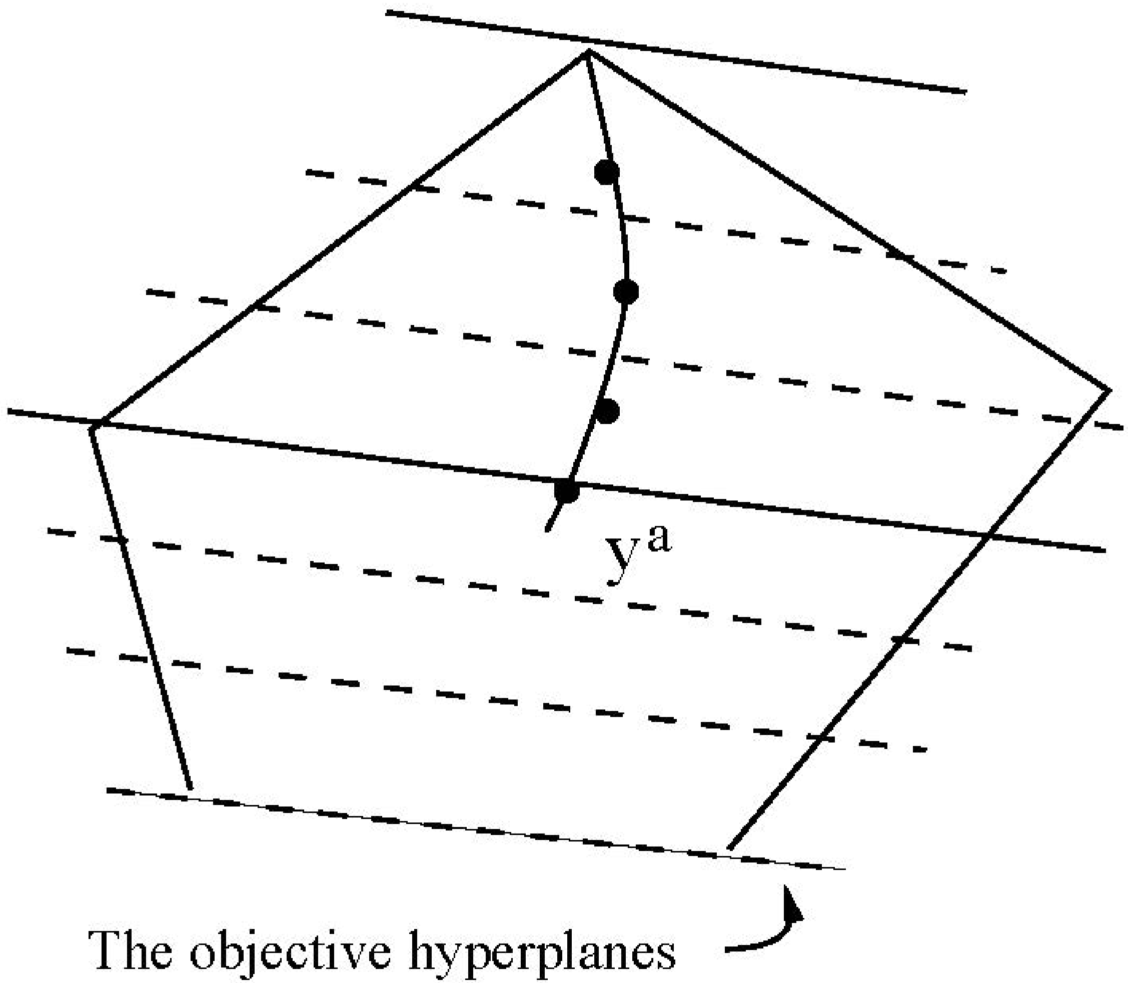

Figure 3: The central path of $\mathbf{y}(\mu)$ in a dual feasible region.

Central Path for Linear Programming

$$\mathcal{C} = \{(\mathbf{x}(\mu), \mathbf{y}(\mu), \mathbf{s}(\mu)) \in \text{int}\,\mathcal{F} : \; X\mathbf{s} = \mu\mathbf{e}, \; 0 < \mu < \infty\};$$

is called the (primal and dual) central path of linear programming.

Theorem 5 *Let both (LP) and (LD) have interior feasible points for the given data set $(A, \mathbf{b}, \mathbf{c})$. Then for any $0 < \mu < \infty$, the central path point pair $(\mathbf{x}(\mu), \mathbf{y}(\mu), \mathbf{s}(\mu))$ exists and is unique. Moreover, the followings hold.*

i) *The central path point $(\mathbf{x}(\mu), \mathbf{s}(\mu))$ is bounded for $0 < \mu \leq \mu^0$ and any given $0 < \mu^0$.*

ii) *For $0 < \mu' < \mu$, $\mathbf{c}^T\mathbf{x}(\mu') < \mathbf{c}^T\mathbf{x}(\mu)$ and $\mathbf{b}^T\mathbf{y}(\mu') > \mathbf{b}^T\mathbf{y}(\mu)$ if both primal and dual have no constant objective values.*

iii) *$(\mathbf{x}(\mu), \mathbf{s}(\mu))$ converges to an optimal solution pair for (LP) and (LD). Moreover, the limit point $\mathbf{x}(0)_{P^*} > \mathbf{0}$ and the limit point $\mathbf{y}(0), \mathbf{s}(0)_{Z^*} > \mathbf{0}$ are the analytic centers of the optimal solution sets of primal and dual, respectively; where (P^*, Z^*) is the strictly complementarity partition if variable index set $\{1, 2, ..., n\}$.*

The Primal-Dual Path-Following Algorithm for LP

In general, we start from an (approximate) central path point $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{s}^k) \in \mathcal{F}$ such that

$$\|X^k \mathbf{s}^k - \mu^k \mathbf{e}\| \le \sigma \mu^k, \quad \text{for some } \sigma \in [0,1).$$

Then, let $\mu^{k+1} = (1-\eta)\mu^k$ for some $\eta \in (0,\ 1]$, we aim to find a new pair $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{F}$ such that $X\mathbf{s} = \mu^{k+1}\mathbf{e}$.

We start from $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{s}^k) \in \mathcal{F}$ and apply the Newton iteration for direction vectors $(\mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_s)$:

$$\begin{array}{rcl} S^k \mathbf{d}_x + X^k \mathbf{d}_s & = & \mu^{k+1}\mathbf{e} - X^k \mathbf{s}^k \\ A\mathbf{d}_x & = & \mathbf{0} \\ A^T \mathbf{d}_y + \mathbf{d}_s & = & \mathbf{0} \end{array},$$

then let $\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}_x, \quad \mathbf{y}^{k+1} = \mathbf{y}^k + \mathbf{d}_y, \quad \mathbf{s}^{k+1} = \mathbf{s}^k + \mathbf{d}_s$. Carefully choosing $\sigma = O(1)$ and $\eta = O(\frac{1}{\sqrt{n}})$ guarantees $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) > \mathbf{0}$ and

$$\|X^{k+1}\mathbf{s}^{k+1} - \mu^{k+1}\mathbf{e}\| \le \sigma \mu^{k+1}, \quad \text{for the same } \sigma \in [0,1).$$

Too many restrictions when following a path... Is a function-driven interior-point algorithm?

Primal-Dual Potential Function for LP

For $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \text{int}\,\mathcal{F}$, the joint primal-dual potential function is defined by

$$\psi_{n+\rho}(\mathbf{x}, \mathbf{s}) := (n+\rho)\log(\mathbf{x}^T\mathbf{s}) - \sum_{j=1}^{n} \log(x_j s_j), \quad \text{for some } \rho > 0.$$

$$\psi_{n+\rho}(\mathbf{x}, \mathbf{s}) = \rho\log(\mathbf{x}^T\mathbf{s}) + \psi_n(\mathbf{x}, \mathbf{s}) \geq \rho\log(\mathbf{x}^T\mathbf{s}) + n\log n,$$

then, for $\rho > 0$, $\psi_{n+\rho}(\mathbf{x}, \mathbf{s}) \to -\infty$ implies that $\mathbf{x}^T\mathbf{s} \to 0$. More precisely, we have

$$\mathbf{x}^T\mathbf{s} \leq \exp\left(\frac{\psi_{n+\rho}(\mathbf{x}, \mathbf{s}) - n\log n}{\rho}\right).$$

Given a pair $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{s}^k) \in \text{int}\,\mathcal{F}$, compute direction vectors $(\mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_s)$ from the Newton iteration:

$$\begin{aligned} S^k\mathbf{d}_x + X^k\mathbf{d}_s &= \frac{(\mathbf{x}^k)^T\mathbf{s}^k}{n+\rho}\mathbf{e} - X^k\mathbf{s}^k, \\ A\mathbf{d}_x &= \mathbf{0}, \\ A^T\mathbf{d}_y + \mathbf{d}_s &= \mathbf{0}. \end{aligned} \tag{9}$$

How to solve the equation system efficiently using the block structures?

Block Structure in the KKT System

$$\begin{aligned} S^k \mathbf{d}_x + X^k \mathbf{d}_s &= \mathbf{r}^k, \\ A\mathbf{d}_x &= \mathbf{0}, \\ A^T \mathbf{d}_y + \mathbf{d}_s &= \mathbf{0}. \end{aligned}$$

Scale the first block to: $\mathbf{d}_x + (S^k)^{-1} X^k \mathbf{d}_s = (S^k)^{-1} \mathbf{r}^k$.

Multiplying A to both sides and using the second block equations: $A(S^k)^{-1} X^k \mathbf{d}_s = A(S^k)^{-1} \mathbf{r}^k$.

Applying the third block equations: $-A(S^k)^{-1} X^k A^T \mathbf{d}_y = A(S^k)^{-1} \mathbf{r}^k$.

This is an $m \times m$ positive definite system, and solve it for $\mathbf{d}_y$; then $\mathbf{d}_s$ from the third block; then $\mathbf{d}_x$ from the first block.

Matrix Factorization to solve $Q\mathbf{y} = \mathbf{r}$:

- Cholesky: $R^T R = Q$, where R is a Right-Triangle matrix
- $LDL^T = Q$, where L is a Left-Triangle matrix.

Description of Algorithm for LP

Given $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{s}^0) \in \text{int}\,\mathcal{F}$. Set $\rho \geq \sqrt{n}$ and $k := 0$.

While $(\mathbf{x}^k)^T\mathbf{s}^k \geq \epsilon$ **do**

1. Set $(\mathbf{x}, \mathbf{s}) = (\mathbf{x}^k, \mathbf{s}^k)$ and compute $(\mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_s)$ from (12).
2. Let $\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha^k\mathbf{d}_x$, $\mathbf{y}^{k+1} = \mathbf{y}^k + \alpha^k\mathbf{d}_y$, and $\mathbf{s}^{k+1} = \mathbf{s}^k + \alpha^k\mathbf{d}_s$ where

$$\alpha^k = \arg\min_{\alpha \geq 0} \psi_{n+\rho}(\mathbf{x}^k + \alpha\mathbf{d}_x, \mathbf{s}^k + \alpha\mathbf{d}_s).$$

3. Let $k := k + 1$ and return to Step 1.

Theorem 6 *Let $\rho \geq \sqrt{n}$. Then, the potential reduction algorithm generates the (interior) feasible solution sequence $\{\mathbf{x}^k, \mathbf{y}^k, \mathbf{s}^k\}$ such that*

$$\psi_{n+\rho}(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) - \psi_{n+\rho}(\mathbf{x}^k, \mathbf{s}^k) \leq -0.15.$$

Thus, if $\psi_{n+\rho}(\mathbf{x}^0, \mathbf{s}^0) \leq \rho \log((\mathbf{x}^0)^T \mathbf{s}^0) + n \log n$, the algorithm terminates in at most $O(\rho \log((\mathbf{x}^0)^T \mathbf{s}^0/\epsilon))$ iterations with $(\mathbf{x}^k)^T \mathbf{s}^k = \mathbf{c}^T \mathbf{x}^k - \mathbf{b}^T \mathbf{y}^k \leq \epsilon$.

The proof used a key fact: $\mathbf{d}_x^T \mathbf{d}_s = -\mathbf{d}_x^T A^T \mathbf{d}_y = 0$ for the directions. Also

$$\begin{aligned}
(\mathbf{x}^k)^T \mathbf{s}^k & \leq \exp\left(\frac{\psi_{n+\rho}(\mathbf{x}^k, \mathbf{s}^k) - n \log n}{\rho}\right) \\
& \leq \exp\left(\frac{\psi_{n+\rho}(\mathbf{x}^0, \mathbf{s}^0) - n \log n - \rho \log((\mathbf{x}^0)^T \mathbf{s}^0/\epsilon)}{\rho}\right) \\
& \leq \exp\left(\frac{\rho \log(\mathbf{x}^0, \mathbf{s}^0) - \rho \log((\mathbf{x}^0)^T \mathbf{s}^0/\epsilon)}{\rho}\right) \\
& = \exp(\log(\epsilon)) = \epsilon.
\end{aligned}$$

The role of ρ? And more aggressive step size?

(LPpdpath...m and LPpdpotential.m of Chapter 5)

Initialization

- Combining the primal and dual into a single linear feasibility problem, then applying LP algorithms to find a feasible point of the problem. Theoretically, this approach can retain the currently best complexity result.
- The big M method, i.e., add one or more artificial column(s) and/or row(s) and a huge penalty parameter M to force solutions to become feasible during the algorithm.
- Phase I-then-Phase II method, i.e., first try to find a feasible point (and possibly one for the dual problem), and then start to look for an optimal solution if the problem is feasible and bounded.
- Combined Phase I-Phase II method, i.e., approach feasibility and optimality simultaneously. To our knowledge, the "best" complexity of this approach is $O(n \log((\mathbf{x}^0)^T\mathbf{s}^0/\epsilon))$.

Homogeneous and Self-Dual Algorithm

- It solves the linear programming problem without any regularity assumption concerning the existence of optimal, feasible, or interior feasible solutions, while it retains the currently best complexity result
- It can start at any positive primal-dual pair, feasible or infeasible, near the central ray of the positive orthant (cone), and it does not use any big M penalty parameter or lower bound.
- Each iteration solves a system of linear equations whose dimension is almost the same as that solved in the standard (primal-dual) interior-point algorithms.
- If the LP problem has a solution, the algorithm generates a sequence that approaches feasibility and optimality simultaneously; if the problem is infeasible or unbounded, the algorithm will produce an infeasibility certificate for at least one of the primal and dual problems.

Primal-Dual Alternative Systems

Recall that a pair of LP has two alternatives

$$\text{(Solvable)} \quad \begin{array}{rl} A\mathbf{x} - \mathbf{b} & = \mathbf{0} \\ -A^T\mathbf{y} + \mathbf{c} & \geq \mathbf{0}, \\ \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} & = 0, \\ \mathbf{y} \text{ free}, \ \mathbf{x} & \geq \mathbf{0} \end{array} \quad \text{or} \quad \text{(Infeasible)} \quad \begin{array}{rl} A\mathbf{x} & = \mathbf{0} \\ -A^T\mathbf{y} & \geq \mathbf{0}, \\ \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} & > 0, \\ \mathbf{y} \text{ free}, \ \mathbf{x} & \geq \mathbf{0} \end{array}$$

$$(HP) \quad \begin{array}{rl} A\mathbf{x} - \mathbf{b}\tau & = \mathbf{0} \\ -A^T\mathbf{y} + \mathbf{c}\tau & = \mathbf{s} \geq \mathbf{0}, \\ \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} & = \kappa \geq 0, \\ \mathbf{y} \text{ free}, \ (\mathbf{x}; \tau) & \geq \mathbf{0} \end{array}$$

where the two alternatives are:

$$\text{(Solvable)}: \ (\tau > 0, \kappa = 0) \quad \text{or} \quad \text{(Infeasible)}: \ (\tau = 0, \kappa > 0)$$

Let's Find a Feasible Solution of (HP)

Given $\mathbf{x}^0 = \mathbf{e} > \mathbf{0}$, $\mathbf{s}^0 = \mathbf{e} > \mathbf{0}$, and $\mathbf{y}^0 = \mathbf{0}$, we formulate a self-dual LP problem:

$$\begin{array}{lllllll}
(HS-DP) & \text{min} & & & & (n+1)\theta & \\
& \text{s.t.} & & A\mathbf{x} & -\mathbf{b}\tau & +\bar{\mathbf{b}}\theta & = \mathbf{0}, \\
& & -A^T\mathbf{y} & & +\mathbf{c}\tau & -\bar{\mathbf{c}}\theta & \geq \mathbf{0}, \\
& & \mathbf{b}^T\mathbf{y} & -\mathbf{c}^T\mathbf{x} & & +\bar{z}\theta & \geq 0, \\
& & -\bar{\mathbf{b}}^T\mathbf{y} & +\bar{\mathbf{c}}^T\mathbf{x} & -\bar{z}\tau & & = -(n+1), \\
& & \mathbf{y}\ \text{free}, & \mathbf{x} \geq \mathbf{0}, & \tau \geq 0, & \theta\ \text{free}. &
\end{array}$$

Note that $(\mathbf{y} = \mathbf{0},\ \mathbf{x} = \mathbf{e},\ \tau = 1,\ \theta = 1)$ is a strictly feasible point for (HSDP). Moreover, one can show that the constraints imply

$$\mathbf{e}^T x + \mathbf{e}^T s + \tau + \kappa - (n+1)\theta = (n+1),$$

which serves as a normalizing constraint for (HSDP) to prevent the all-zero solution.

Main Result

Theorem 7 *The interior-point algorithm solves (HS-DP) in* $O(\sqrt{n}\log\frac{n}{\epsilon})$ *steps and each step solves a system of linear equations as the same size as in feasible algorithms, and it always produces an optimal solution* $(\mathbf{y}^*, \mathbf{x}^*, \tau^*, \mathbf{s}^*, \kappa^*, \theta^* = 0)$ *where* $\tau^* + \kappa^* > 0$*. If* $\tau^* > 0$ *then it produces an optimal solution pair for the original LP problem; if* $\kappa^* > 0$*, then it produces a certificate to prove (at least) one of the pair is infeasible.*

(HSDLPsolver....m)

Extensions to Solving SDP: Potential Function

For any $X \in \text{int}\, \mathcal{F}_p$ and $(\mathbf{y}, S) \in \text{int}\, \mathcal{F}_d$, let parameter $\rho > 0$ and

$$\psi_{n+\rho}(X, S) := (n+\rho)\log(X \bullet S) - \log(\det(X) \cdot \det(S)),$$

$$\psi_{n+\rho}(X, S) = \rho \log(X \bullet S) + \psi_n(X, S) \geq \rho \log(X \bullet S) + n \log n.$$

Then, $\psi_{n+\rho}(X, S) \to -\infty$ implies that $X \bullet S \to 0$. More precisely, we have

$$X \bullet S \leq \exp(\frac{\psi_{n+\rho}(X, S) - n \log n}{\rho}).$$

Primal-Dual SDP Alternative Systems

A pair of SDP has two alternatives under mild conditions

$$
\text{(Solvable)} \quad
\begin{array}{rl}
\mathcal{A}X - \mathbf{b} & = \mathbf{0} \\
-\mathcal{A}^T\mathbf{y} + C & \succeq \mathbf{0}, \\
\mathbf{b}^T\mathbf{y} - C \bullet X & = 0, \\
\mathbf{y} \text{ free}, \ X & \succeq \mathbf{0}
\end{array}
\quad \text{or} \quad
\text{(Infeasible)} \quad
\begin{array}{rl}
\mathcal{A}X & = \mathbf{0} \\
-\mathcal{A}^T\mathbf{y} & \succeq \mathbf{0}, \\
\mathbf{b}^T\mathbf{y} - C \bullet X & > 0, \\
\mathbf{y} \text{ free}, \ X & \succeq \mathbf{0}
\end{array}
$$

An Integrated Homogeneous System

The two alternative systems can be homogenized as one:

$$
\begin{array}{lrl}
(HSDP) & \mathcal{A}X - \mathbf{b}\tau & = \mathbf{0} \\
& -\mathcal{A}^T\mathbf{y} + C\tau & = \mathbf{s} \geq \mathbf{0}, \\
& \mathbf{b}^T\mathbf{y} - C \bullet X & = \kappa \geq 0, \\
& \mathbf{y} \text{ free}, \;\; X \succeq \mathbf{0}, & \tau \geq 0,
\end{array}
$$

where the three alternatives are

$$
\begin{array}{rl}
\text{(Solvable)}: & (\tau > 0, \kappa = 0) \\
\text{(Infeasible)}: & (\tau = 0, \kappa > 0) \\
\text{(All others)}: & (\tau = \kappa = 0).
\end{array}
$$

Primal-Dual Interior-Point Algorithms for General Convex Optimization I

We now present an algorithm for solving more general convex optimization problems:

$$\text{min} \;\; f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \geq \mathbf{0}$$

where, with the notation $X = \text{diag}(\mathbf{x})$, we look for a constrained root of

$$X\mathbf{g}(\mathbf{x}) = \mathbf{0}, \; \mathbf{x} \geq \mathbf{0}, \; \mathbf{g}(\mathbf{x}) \geq \mathbf{0}, \quad \text{where} \quad \mathbf{g}(\mathbf{x}) = \nabla f(\mathbf{x}).$$

Vector function $\mathbf{g}(\mathbf{x})$ would be a monotone mapping and the solution is also called the monotone complemtarity point.

We assume that f meets a Scaled Lipschitz condition: for any point $\mathbf{x} > \mathbf{0}$

$$\|X\left(\mathbf{g}(\mathbf{x}+\mathbf{d}) - \mathbf{g}(\mathbf{x}) - \nabla\mathbf{g}(\mathbf{x})\mathbf{d}\right)\| \leq \beta_\alpha \mathbf{d}^T \nabla\mathbf{g}(\mathbf{x})\mathbf{d}, \; \text{whenever} \; \|X^{-1}\mathbf{d}\| \leq \alpha(< 1). \qquad (10)$$

Examples of Scaled Lipschitz Functions

- $f(x) = -\log(x)$, $g(x) = \frac{-1}{x}$ and $g'(x) = \frac{1}{x^2}$: Not Lipschitz but Scaled Lipschitz

$$\frac{-1}{x+d} - \frac{-1}{x} - \frac{d}{x^2} = \frac{1}{x}\left(\sum_{p=2}^{\infty}(\frac{-d}{x})^p\right) \le \frac{d^2}{x^3}\frac{1}{1-\alpha} \Rightarrow \beta_\alpha = \frac{1}{1-\alpha}.$$

- $f(x) = x\log(x)$, $g(x) = 1 + \log(x)$ and $g'(x) = \frac{1}{x}$: Not Lipschitz but Scaled Lipschitz.
- $f(x) = e^x$, $g(x) = e^x$ and e^x: Both Lipschitz and Scaled Lipschitz at Bounded x.

Interior-Point Algorithms for General Convex Optimization II

We start from a solution $\mathbf{x}^k > \mathbf{0}$ and $\mathbf{g}(\mathbf{x}^k) > \mathbf{0}$ and they approximately satisfy the equations

$$X\mathbf{s} = \mu^k \mathbf{e}, \quad \mathbf{s} = \mathbf{g}(\mathbf{x}), \quad \text{for some} \quad \mu^k > 0. \tag{11}$$

Such a solution exists because it is the (unique) optimal solution for the problem with logarithmic barrier

$$\text{min} \;\; f(\mathbf{x}) - \mu^k \sum_j \log(x_j).$$

We replace μ^k by $\mu^{k+1} = (1 - \frac{\eta}{\sqrt{n}})\mu^k$ and aim to find a solution $\mathbf{x} > \mathbf{0}$ such that $\mathbf{g}(\mathbf{x}) > \mathbf{0}$

$$X\mathbf{s} = \mu^{k+1}\mathbf{e}, \quad \mathbf{s} = \mathbf{g}(\mathbf{x}).$$

Starting from $(\mathbf{x}^k, \mathbf{s}^k)$, we apply the Newton iteration using the auxiliary variables $\mathbf{s} = \mathbf{g}(\mathbf{x})$ ($\mathbf{s}^k = \mathbf{g}(\mathbf{x}^k)$):

$$\begin{aligned} X^k \mathbf{d}_s + S^k \mathbf{d}_x &= (1 - \tfrac{\eta}{\sqrt{n}})\mu^k \mathbf{e} - X^k \mathbf{s}^k, \\ \mathbf{d}_s &= \nabla \mathbf{g}(\mathbf{x}^k)\mathbf{d}_x. \end{aligned} \tag{12}$$

One can analyze the quality of the new iterate $\mathbf{x}^+ = \mathbf{x}^k + \mathbf{d}_x$ together with $\mathbf{s}^+ = \mathbf{g}(\mathbf{x}^+)$ when the operator $\mathbf{g}(\mathbf{x}^k)$ is monotonic.

There is also a Homogeneous and Self-Dual Algorithm for solving the monotone complementarity problem, which is a basic solver of MOSEK. The algorithm produces a certificate if no complementarity solution exists. (HOmcp.m, mcpfun.m and mcpJacobian.m of Chapter 15)

Software Implementation

Cplex-Barrier IBM, GUROBI, COPT

SEDUMI: http://sedumi.mcmaster.ca/

MOSEK: http://www.mosek.com/products_mosek.html

SDDPT3: http://www.math.nus.edu.sg/~mattohkc/sdpt3.html

DSDP (Dual Semidefinite Programming Algorithm): http://www.stanford.edu/~yyye/Col.html

CVX/ECOS: http://www.stanford.edu/~boyd/cvx

hsdLPsolver and more: http://www.stanford.edu/~yyye/matlab.html