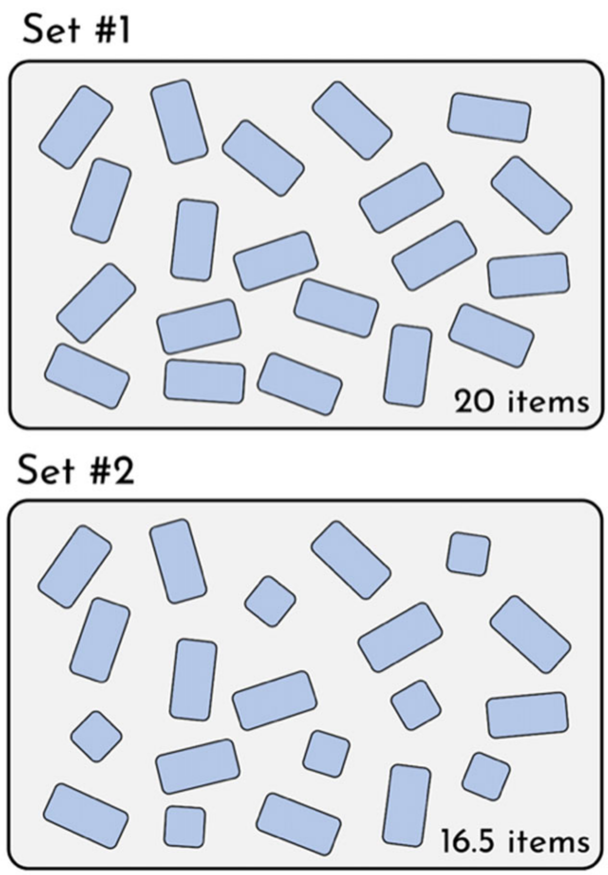

Figure 1 (Yousif). Two sets, one with 20 "full" objects, and one with 16.5 "full" objects. Does the number system represent these partial objects?

the number system is just representing the fact that there are partial entities in the display? What if the number system is representing the rational number 16.5?

If true, this leads to the novel prediction above: that area ought to influence number perception (insofar as area reflects partial wholes). After all, spatial cues influence the perception objecthood (Franconeri, Bemis, & Alvarez, 2009; Yu, Xiao, Bemis, & Franconeri, 2019), and the visual system must be counting objects. This would be ecologically realistic. Seven half meals are equivalent to 3.5 full meals; it could be argued that this latter quantity, and not the former, is the better one to represent. (At the extremes, this would not be viable. One massive object probably ought not to be equated with 100 tiny objects of equivalent size. But, under most circumstances, similar objects are often of a fairly similar size. Edges cases such as these likely would not occur very often in the natural environment. Or, maybe they do! My suggestion is only that future research could consider this intriguing possibility.)

This possibility may or may not pan out empirically, yet it is one of many ways that Clarke and Beck's suggestion could radically alter how we think about quantity perception moving forward.

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Constructing rationals through conjoint measurement of numerator and denominator as approximate integer magnitudes in tradeoff relations

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Abstract

To investigate mechanisms of rational representation, I consider (1) construction of an ordered continuum of psychophysical scale of magnitude of sensation; (2) counting mechanism leading to an approximate numerosity scale for integers; and (3) conjoint measurement structure pitting the denominator against the numerator in tradeoff positions. Number sense of resulting rationals is neither intuitive nor expedient in their manipulation.

The proposal by Clarke and Beck (C&B) that the approximate number system (ANS) represents rational numbers is specific and intriguing. In this comment, I shall speculate on representational mechanisms through which rationals are deployed by the mind as a numerical representation of magnitude continuum.

Representation of magnitude of sensations was long understood, through the ideas of "just-noticeable difference" (Weber fraction) and Fechner's logarithmic scaling of sensation in relation to the intensity of eliciting stimuli. With a series of psychophysical experiments using a variety of direct-scaling paradigms (e.g., magnitude estimation, magnitude production, and cross-modality matching), Stevens (Stevens, 1936, 1956; Stevens & Galanter, 1957) unequivocally established this psychophysical scale as an ordered continuum admitting concatenation operation, and proposed the power-law to map physical intensities to a psychological scale (Stevens, 1957). Fechner's logarithmic scaling and Stevens' power scaling can be reconciled as the former can be taken as the extreme case of the latter, when the exponent β approaches zero:

$$\frac{1}{\beta}(I^{\beta} - 1) \rightarrow \log I.$$

When it comes to numerical representation of the magnitude scale, things become subtler because of co-existing, symbolic and non-symbolic aspects of numerals (Feigenson, Carey, & Spelke, 2002; Feigenson, Dehaene, & Spelke, 2004; Lemer, Dehaene, Spelke, & Cohen, 2003; Xu, Spelke, & Goddard, 2005). Numerosity as the pre-linguistic faculty to represent numerical information, possibly congruent with the psychophysical scale mentioned above, is clearly different from counting (Carey, 2001, 2009; Le Corre & Carey, 2007; Sarnecka & Carey, 2008), which involves the mastery of an array of cognitive routines such as a "successor" function, one-to-one correspondence, and "cardinality" and "equality" of sets for establishing the concept of integer (Carey & Barner, 2019). Whether or not subitization (the process of immediate and accurate recognition of small numbers) plays any role for enabling the counting routine, the resulting system of integers acquires an approximate numerosity scale, and maps linearly or logarithmically to the psychophysical scale (Dehaene, Izard, Spelke, & Pica, 2008; Núñez, Cooperrider, & Wassmann, 2012). This is possible because both counting (which is anchored on "successor function") and numerosity estimation (which is subject to Weber's Law) are predicated on individuation of objects.

The authors referenced the work of He, Zhang, Zhou, and Chen (2009). In that paper, participants were briefly presented with visual displays of dots in random positions and were asked to judge their numerosities; the brevity of stimulus presentations precluded any counting strategies. In some displays, pairs of adjacent dots were connected by line segments whereas in others, line segments were freely hanging without touching the dots (see the reproduced Fig. 1d of C&B). The line segments were introduced to manipulate object individuation aspect of numerosity estimation. Results clearly showed that connecting the pairs of dots by line segments led to an underestimation of dot numbers in those patterns because of decreased individuation. Thus, we suggested "two different stages underlying numerosity perception: first, the individuation of items in a visual display, followed by magnitude estimation based on the distinct number of items" (p. 517).

Numerosity estimation mechanism (approximate integer scale) mentioned above is, of course, congruent with the psychophysical scale revealed in Steven's direct-scaling experiments. In the language of axiomatic foundation of measurement (FoM) (Krantz, Luce, Suppes, & Tversky, 1971), the common underlying measurement structure is that of ordered concatenation group. Their difference is that numerosity scale is countable, and hence the group of integers ($\mathbf{Z}$, +), whereas the psychophysical scale is uncountable, and hence the group of reals ($\mathbf{R}$, +) which is isomorphic to the group of positive reals ($\mathbf{R}_{+}$, $\times$). Note that in any ordered concatenation structure (M, $\oplus$), there is only one binary operator, $\oplus$, on the set M (which can be $\mathbf{R}$, $\mathbf{R}_{+}$, or $\mathbf{Z}$) of elements whose magnitude are in total "order." This binary operator $\oplus$, takes in two elements of the set and outputs another element larger than any of its inputs. That ($\mathbf{R}$, +) and ($\mathbf{R}_{+}$, $\times$) are mathematically isomorphic can be easily seen by the identity: $\log(a \times b) = \log a + \log b$. Cohen and Narens (Cohen & Narens, 1979; Narens, 1981) developed the elaborative theory of ratio-scalability for extensive concatenation systems and the associated scale types.

The magnitude system conceptualized as above does *not* automatically come with a multiplication or $\otimes$ operation that accompanies (and distributes across) the $\oplus$ operation. That is, the magnitude system is not an algebra with two interwoven operations. To remedy this in FoM, Narens (Luce & Narens, 1987; Narens & Luce, 1986) proposed to use the automorphism group on the representational space, either ($\mathbf{R}$, +) or ($\mathbf{R}_{+}$, $\times$), as a surrogate of $\otimes$. An automorphism σ of a measurement structure is a bijective transformation from the structure to itself (i.e., mapping one element to another) that preserves the order relationship among its elements:

$$\sigma(a \oplus b) = \sigma(a) \oplus \sigma(b), \quad a > b \text{ iff } \sigma(a) > \sigma(b).$$

All automorphisms form (generally non-commutative) group, with group multiplication operation $\otimes$ implemented as successive application of two automorphisms: $\sigma_2(\sigma_1(a)) = (\sigma_2 \otimes \sigma_1)(a)$. Distributivity of $\otimes$ over $\oplus$ always holds, but $\otimes$ is, in general, a non-commutative multiplication.

In FoM (Krantz et al., 1971), commutative multiplication $\otimes$ of two elements is through the construction of a conjoint measurement structure. This is the structure involved in tradeoff of Utility and Risk in Value = (Utility, Risk), and Length and Width in Area = (Length, Width). In the current case, we have $\mathbf{Q} = (D, N)$ where D is the denominator and N the numerator of a rational number $\mathbf{Q}$. Here, both (D, +) and (N, +) are integer or numerosity structure ($\mathbf{Z}$, +).

Additive conjoint structure is axiomatized by various cancellation conditions across its two underlying "components." Essential to an additive conjoint measurement is the independence assumptions about its components and the existence of tradeoff, or the "indifference curve" of equal value.

When the rational field $\mathbf{Q} = (D, N)$ is constructed this way, addition and multiplication of two rationals $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be quite convoluted. Therefore, achieving "dense" representation of magnitude scale using rationals, as C&B suggested, has a heavy price toll when computing the fraction addition and fraction multiplication using only the routines for (D, +) and (N, +). Interference effects should readily be expected.

To summarize, I endorse the proposal of C&B to construct ANS by rationals, and amend it with a suggestion that such representation is achieved through conjoint measurement and tradeoff between the approximate integer (numerosity) representations of the denominator and the numerator.

Conflict of interest. None.

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Authors' Response

Numbers, numerosities, and new directions

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Abstract

In our target article, we argued that the number sense represents natural and rational numbers. Here, we respond to the 26 commentaries we received, highlighting new directions for empirical and theoretical research. We discuss two background assumptions, arguments against the number sense, whether the approximate number system (ANS) represents numbers or numerosities, and why the ANS represents rational (but not irrational) numbers.

We are humbled to have received 26 commentaries from 62 researchers, among them many of our academic heroes. Unsurprisingly, these commentaries reflect a diversity of opinion. Some endorse and build upon the main conclusions of our paper; others highlight points of disagreement. Although we remain confident in our main theses, we learned a great deal from our commentators – about soft spots in our arguments, points that require development, where we could have been clearer, and avenues for future research. We're extremely grateful for their insights.

Our replies follow the order of our target article. We discuss two background assumptions, arguments against the number sense, whether the approximate number system (ANS) represents numbers or numerosities, and why the ANS represents rational (but not irrational) numbers.

R1. Background assumptions

Explanation needs to start somewhere, and our discussion presupposed that the ANS is representational and that it sometimes operates in perception, enabling numbers to enter perceptual contents. Some commentators challenged these background assumptions.

R1.1. Is the ANS representational?

While the idea that the ANS *represents* anything at all is relatively uncontroversial among ANS researchers (but see Beck [2015] for a defense), **Jones, Zahidi, and Hutto (Jones et al.)** suggest that our commitment to representations imports unnecessary "philosophical baggage." They recommend instead embracing an anti-representational *Radical Enactivism*.

In general, we're dubious when people tell us we can avoid philosophical baggage by embracing views with "radical" in the name. **Jones et al.**'s "radical" vision is that we acquire a perceptual sensitivity to numbers simply by virtue of our sensitivity to the affordances they enable: "The 'sevenness' is not a property of the apples, nor of the perceiver, but of what the perceiver can do with them." The trouble is: It's essentially open ended what you can do when you perceive there to be seven of something. Therefore, we don't see how the perception of number can be specified in these terms. Furthermore, representation is fundamental to explanations of the ANS's internal computations. For instance, when children use their ANS to add the number of blue dots and red dots in a sequence of events (e.g., Barth et al., 2005), it's not just that they're afforded with (say) the sevenness of the blue dots, the tenness of the red dots, and then magically afforded with the seventeenness of the red and blue dots; they engage in a computational transition, in which internal states of the organism interact in content respecting ways. This presupposes representation.

R1.2. Are numbers perceivable?

Aulet and Lourenco, **Marshall**, **Novaes and dos Santos**, and **Opfer, Samuels, Shapiro, and Snyder** (**Opfer et al.**) all questioned our assumption that numbers are perceivable.

Because numbers are higher-order, **Novaes and dos Santos** and **Marshall** think they cannot be perceived. Marshall proclaims that "second-order entities are no part of the sensible realm," while Novaes and dos Santos write that numbers "emerge only after an agent has adopted a given sortal" and thus are not "out there, inexact or otherwise, to be represented." But we find this puzzling. Surely, it's an objective fact that the apples on my kitchen counter total five in number. This fact is "out there" and does not require anyone's mind to "emerge." It's also the sort of thing one should expect perceptual systems to be capable of picking up on. We're not sure why anyone would think otherwise unless they were committed to an outdated view of perception according to which perception only represents properties for which we have dedicated sensory transducers. But perception is not sensation. At least since Helmholtz, we've known that