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Chapter 3
Affine Connections with Torsion in (Para-)complexified Structures

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Abstract We investigate integrability conditions for an almost (para-)complex structure L, $L^2 = \pm\text{id}$, on manifolds that admit affine connections carrying torsion in general. The affine connections ∇ under consideration are not assumed to be (para-)complex $\nabla L \neq 0$. Two kinds of torsion matching conditions leading to integrability of L are analyzed, which are generalizations of the first and second canonical connections along the Gauduchon line. We also discuss (para-)holomorphicity of ∇ encoded by the second condition.

3.1 Introduction

On a differentiable manifold $\mathcal{M}$ of even dimension, one can separately consider three entities all taking inputs from sections $\Xi(T\mathcal{M})$ of the tangent bundle $T\mathcal{M}$—an affine connection $\nabla : \Xi(T\mathcal{M}) \times \Xi(T\mathcal{M}) \to \Xi(T\mathcal{M})$, a pseudo-Riemannian metric $g : \Xi(T\mathcal{M}) \times \Xi(T\mathcal{M}) \to C\mathcal{M}$, and a tangent bundle endomorphism-induced operator $L : \Xi(T\mathcal{M}) \to \Xi(T\mathcal{M})$. Here and below, we use L to denote either an *almost complex* operator J, $J^2 = -\text{id}$, or an *almost para-complex* operator K, $K^2 = \text{id}$ assuming the multiplicity of the $\pm$ 1 eigenvalues are equal. We use the term "(para-) complex" to mean "either complex or para-complex", corresponding to when $L = J$ or when $L = K$.

Enforcing compatibility of L with g

$$g(LX, Y) + g(X, LY) = 0, \tag{3.1}$$

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for any vector fields X, Y on $\mathcal{M}$, that is, $X, Y \in \Xi(T\mathcal{M})$, leads to an almost (para-) Hermitian structure of $(\mathcal{M}, g, L)$. Here, g is positive-definite and hence a Riemannian metric when $L = J$, and is of split signature and hence a Norden metric when $L = K$. When L is integrable, $(\mathcal{M}, g, L)$ becomes a *(para-)Hermitian manifold*. Compatibility of g and ∇ has been investigated in the context of affine differential geometry and information geometry, through the notion of "Codazzi-coupling" (definition to follow). A manifold $\mathcal{M}$ equipped with a metric g and a torsion-free connection ∇ is called a *statistical manifold* if (g, ∇) is Codazzi-coupled [1]. Finally, compatibility of L with (a torsion-free) ∇ has also been studied, in terms of Codazzi-coupling between them (e.g. [2, 3]). In [4], it was further shown that when a torsion-free connection ∇ is Codazzi-coupled both to g and to L, then $(\mathcal{M}, g, L)$ in fact turns into a (para-)Kähler manifold. Such a (para-)Kähler manifold is called *Codazzi-(para-)Kähler* manifold [4] because of the additional structure imposed by ∇. A special case is the class of "special Kähler manifold" of [2], with ∇ assumed to be curvature-free in addition to being torsion-free.

Given (g, L) on $\mathcal{M}$, one can define g-conjugate transformation (denoted ∇^*) and L-gauge transformation (denoted ∇^L), respectively, given any connection ∇ on $\mathcal{M}$. These transformations are involutive $(\nabla^*)^* = \nabla = (\nabla^L)^L$. Details were in [4] and will be given below. When Eq. (3.1) is satisfied, then these two transformations of ∇ are commutative $(\nabla^*)^L = (\nabla^L)^* \equiv \nabla^\dagger$, so that $\{\mathrm{id}, *, L, \dagger\}$ form a 4-element Klein group of transformation of affine connections on $\mathcal{M}$ (Theorem 2.13 of [4]). Under this scenario, $\nabla^\dagger$ then becomes the conjugate connection of ∇ with respect to the fundamental form ω defined by $\omega(X, Y) = g(LX, Y)$; here (g, L, ω) is known as the "compatible triple."

How this quadruple of connections $(\nabla, \nabla^*, \nabla^L, \nabla^\dagger)$ interact with g and L on an almost (para-)Hermitian manifold $(\mathcal{M}, g, L)$ deserves further investigation. Since Codazzi-coupling of ∇ with either g or L ensures that ∇^* or ∇^L will have same torsion as that of ∇, enforcing Codazzi couplings of a torsion-free ∇ *both* with g *and* with L will lead to vanishing torsion for the entire quadruple of connections $(\nabla, \nabla^*, \nabla^L, \nabla^\dagger)$. In this case (see [4]) $\nabla^\dagger = \nabla$, so Codazzi-(para-)Kähler manifolds admit pairs of torsion-free connections. These manifolds are both *(para-)Kähler* manifolds (with integrable L and d-closed ω) and *statistical* manifolds (with a family of α-connections [5]). Manifolds where $\nabla^\dagger = \nabla$, or equivalently $\nabla\omega = 0$, are called *holomorphic statistical manifold* [6] (a notion originally due to Takashi Kurose) when torsion of ∇ is zero; in this case, ∇ is a symplectic connection. So Codazzi-(para-)Kähler manifolds are important examples of holomorphic statistical manifolds.

In a recent paper, Grigorian and Zhang [7] relaxed the restriction of torsion-freeness of ∇, and investigated integrable structures on an almost (para-)Hermitian manifold $(\mathcal{M}, g, L)$ that nevertheless admits affine connections with torsion. Connections with torsion on Hermitian manifolds have always been a topic of interest, e.g. [8, 9]. The work of [4, 7] investigated connections that are not necessarily parallel to L, $\nabla L \neq 0$. That is, neither ∇ nor ∇^L is (para-)complex, though $\frac{1}{2}(\nabla + \nabla^L)$ is always parallel to L:

$$\frac{1}{2}(\nabla + \nabla^L)L = 0.$$

Along with $\nabla^L := L^{-1}\nabla L$, the L-conjugate transformation of ∇, we can define *torsion-coupling* of (∇, L) as the following relation (Definition 1) between the torsions T^∇ and T^{∇^L} of ∇ and ∇^L:

$$T^\nabla(LX, Y) = L(T^{\nabla^L}(X, Y)). \tag{3.2}$$

Torsion-coupling is equivalent to Codazzi-coupling when torsion of ∇ vanishes. So torsion-coupling of ∇ with L provides an appropriate generalization to impose Codazzi-like coupling requirement for connections with non-vanishing torsion. It is shown ([7], Theorem 6) that ∇ is torsion-coupled to L if any only L is integrable *and* ∇ is (para-)holomorphic. Moreover, given a (para-)Hermitian manifold $(\mathcal{M}, g, L)$ with four connections $\nabla, \nabla^L, \nabla^*, \nabla^\dagger \equiv \nabla^{*L} = \nabla^{L*}$, all possibly carrying torsion, if any of the four is torsion-coupled to L and any of the four is Codazzi-coupled to g, then *all* four must be torsion-coupled to L *and* Codazzi-coupled to g ([7], Theorem 12). This leads to the definition of a *Codazzi-(para-)Hermitian* structure–which is a (para-)Hermitian manifold with all four of the aforementioned connections being (para-)holomorphic and Codazzi-coupled to g. In this case, their torsions satisfy $T = T^*$, $T^L = T^\dagger$, but $d\omega \neq 0$ in general, unless the torsions of the quadruple of the connections are all zero.

The torsion-coupling condition Eq. (3.2) on an almost (para-)complex manifold $(\mathcal{M}, L)$, as first investigated in [7], is interesting because the same equation (called the *2nd Matching Condition* or MC2 below) encodes two things simulataneously, both a constraint on L, i.e., L should be integrable, and a constraint on ∇, i.e., ∇ should be (para-)holomorphic. In this paper, we investigate this torsion-coupling condition further—by decomposing this condition into two parts, one which implies integrability of L, and another which implies (para-)holomorphicity of ∇. We investigate another coupling of L with ∇ (called the *1st Matching Condition* or MC1 below) that would lead to integrability condition of L and certain properties of ∇ in reference to the Gauduchon line [8]. Our goal in this paper is to understand how torsion of a connection interacts with a (para-)complex structure L on a manifold when the connection is (in general) not required to be parallel with respect to L.

3.2 Torsion of ∇ and Integrability of L

3.2.1 L Conjugation of ∇

Given a real manifold $\mathcal{M}$ of even dimensions $\dim \mathcal{M} = 2n$, we study affine connection ∇ (in general carrying torsion and curvature) on $\mathcal{M}$, i.e., ∇ operates on sections $\Xi(T\mathcal{M})$ of its tangent bundle $T\mathcal{M}$, i.e., vector fields with real dimension $2n$. Let $L : \Xi(T\mathcal{M}) \to \Xi(T\mathcal{M})$ be an a map induced by tangent bundle endomorphism, with L^{-1} denoting the inverse map. Then, the L-gauge transformed connection is defined as $\nabla^L := L^{-1} \circ \nabla \circ L$, or

$$\nabla^L_X Y = L^{-1}(\nabla_X(LY)) \tag{3.3}$$

for any vector fields X and Y on $\mathcal{M}$, i.e., X and Y are elements of $\Xi(T\mathcal{M})$. That ∇^L is indeed an affine connection needs verification, which we omit here. The L-gauge transformation $\nabla \to \nabla^L$ of any connection ∇ satisfies

$$(\nabla^{L_1})^{L_2} = \nabla^{L_1 \circ L_2},$$

with $\circ$ denoting the composition of L_1 and L_2 as operators. Identifying $\circ$ as the group multiplication operation and the identity transformation $L = \mathrm{id}$ as the group identity element, L-gauge transformation of affine connections form a transformation group acting on the space of linear connections on $\mathcal{M}$.

It can be seen that $\nabla = \nabla^L$ if and only if $\nabla L = 0$. A connection ∇ is called *(para-)complex* if $\nabla L = 0$. In this paper, we do not generally make the assumption of (para-)complexity of ∇.

We define a vector-valued skew-symmetric bilinear form S, a $(1, 2)$-tensor, via the expression

$$S(X, Y) = (\nabla_X L)Y - (\nabla_Y L)X, \tag{3.4}$$

where

$$(\nabla_X L)Y = \nabla_X(LY) - L(\nabla_X Y).$$

The pair (∇, L) is said to be *Codazzi-coupled* if $S = 0$.

Recall that the torsion T^∇ of ∇ is defined as:

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{3.5}$$

We easily derive the identify

$$S(X, Y) = L(T^{\nabla^L}(X, Y) - T^\nabla(X, Y)). \tag{3.6}$$

This leads to the following well-known results.

Proposition 1 *(e.g., [10]) Let ∇ be an affine connection, and let L be an arbitrary tangent bundle isomorphism. Then the following statements are equivalent:*

(i) (∇, L) *is Codazzi-coupled.*
(ii) $T^\nabla(X, Y) = T^{\nabla^L}(X, Y)$.
(iii) (∇^L, L^{-1}) *is Codazzi-coupled.*

Corollary 1 *For the special case of (para-)complex operators* $L^2 = \pm\mathrm{id}$,

(a) $\nabla^L = \nabla^{L^{-1}}$, *i.e., L-conjugate transformation is involutive,* $(\nabla^L)^L = \nabla$.
(b) (∇, L) *is Codazzi-coupled if and only if* (∇^L, L) *is Codazzi-coupled.*

Codazzi-coupling of (∇, L) on an almost-(para-)complex manifold $(\mathcal{M}, L)$ mirrors Codazzi-coupling of (∇, g) on a Riemannian manifold $(\mathcal{M}, g)$.

3.2.2 Nijenhuis Tensor N_L and Integrability

On an (even-dimensional) manifold $\mathcal{M}$ with an almost (para-)complex structure L, there is an associated Nijenhuis tensor

$$N_L(X, Y) = -L^2[X, Y] + L[X, LY] + L[LX, Y] - [LX, LY]. \tag{3.7}$$

When $N_L = 0$, the operator L is said to be integrable. Via the celebrated Newlander-Nirenberg theorem, in the complex case $L = J$, this is the obstruction to turning an almost complex structure J into a complex structure. In other words, vanishing of N_J is equivalent to $\mathcal{M}$ admitting local holomorphic charts, with J as the complex structure.

The following identity for the Nijenhuis tensor can be easily verified.

Lemma 1 *The Nijenhuis tensor N_L satisfies*

$$N(LX, Y) = N(X, LY) = -LN(X, Y).$$

Proof Because $L^2 = \pm\mathrm{id}$, it can be taken out of the bracket,

$$\begin{aligned}
N(LX, Y) &= -L^2[LX, Y] + L[LX, LY] + L[L^2X, Y] - [L^2X, LY] \\
&= -L^2[LX, Y] + L[LX, LY] + L^3[X, Y] - L^2[X, LY] \\
&= -L(L[LX, Y] - [LX, LY] - L^2[X, Y] + L[X, LY]) \\
&= -LN(X, Y).
\end{aligned}$$

Likewise, $N(X, LY) = -LN(X, Y)$.

Proposition 2 *The Nijenhuis tensor N_L can be expressed in the follow forms:*

$$N_L(X, Y) = \left(T^{\nabla^L}(L(LX), Y) - LT^{\nabla}(LX, Y)\right) + \left(T^{\nabla^L}(LX, LY) - LT^{\nabla}(X, LY)\right) \tag{3.8}$$

and

$$N_L(X, Y) = \left(T^{\nabla^L}(L(LX), Y) + T^{\nabla^L}(LX, LY)\right) - L\left(T^{\nabla}(LX, Y) + T^{\nabla}(X, LY)\right). \tag{3.9}$$

Proof The relationships were derived in Lemma 3 [7]. We prove them below (to correct certain typos there).

Starting from Eqs. (3.4), (3.5), and (3.6), we have

$$\begin{aligned}
\nabla_X(LY) - \nabla_Y(LX) &= (\nabla_X L)Y + L(\nabla_X Y) - ((\nabla_Y L)X + L(\nabla_Y X)) \\
&= S(X, Y) + L(\nabla_X Y - \nabla_Y X) \\
&= S(X, Y) + L([X, Y] + T^{\nabla}(X, Y)) \\
&= LT^{\nabla^L}(X, Y) + L[X, Y].
\end{aligned} \tag{3.10}$$

Substituting $X \to LX$, $Y \to LY$ in Eq. (3.10) yields

$$\nabla_{LX}(L^2Y) - \nabla_{LY}(L^2X) = LT^{\nabla^L}(LX, LY) + L[LX, LY],$$

or, after cancelling an L on both sides

$$L(\nabla_{LX}Y - \nabla_{LY}X) = T^{\nabla^L}(LX, LY) + [LX, LY].$$

Applying L operator on both sides of Eq. (3.10) and adding to the above, we have

$$\begin{aligned} &L(\nabla_X(LY) - \nabla_Y(LX) + \nabla_{LX}Y - \nabla_{LY}X) \\ &= L^2\, T^{\nabla^L}(X, Y) + L^2[X, Y] + T^{\nabla^L}(LX, LY) + [LX, LY], \end{aligned}$$

or

$$\begin{aligned} &L([LX, Y] + T^{\nabla}(LX, Y) + [X, LY] + T^{\nabla}(X, LY)) \\ &= L^2\, T^{\nabla^L}(X, Y) + T^{\nabla^L}(LX, LY) + L^2[X, Y] + [LX, LY]. \end{aligned}$$

Rearranging the terms yields

$$\begin{aligned} &-L^2[X, Y] + L[LX, Y] + L[X, LY] - [LX, LY] \\ &= L^2\, T^{\nabla^L}(X, Y) + T^{\nabla^L}(LX, LY) - LT^{\nabla}(LX, Y) - LT^{\nabla}(X, LY). \end{aligned}$$

The lefthand side is nothing but $N_L(X, Y)$. So we obtain Eqs. (3.8) and (3.9).

3.2.3 MC1 Versus MC2

A careful examination of expressions (3.8) and (3.9) reveals two different scenarios under which N_L vanishes—these scenarios will be referred to as *1st Matching Condition* (MC1) and *2nd Matching Condition* (MC2) of (∇, L).

Definition 1 A pair (∇, L) is said to satisfy

(i) *1st Matching Condition*, or MC1, if the following holds:

$$T^{\nabla}(LX, Y) + T^{\nabla}(X, LY) = 0; \tag{3.11}$$

(ii) *2nd Matching Condition*, or MC2, if the following holds:

$$T^{\nabla}(LX, Y) = LT^{\nabla}(X, Y) + (\nabla_X L)Y - (\nabla_Y L)X, \tag{3.12}$$

or equivalently,

$$T^{\nabla}(LX, Y) = L(T^{\nabla^L}(X, Y)). \tag{3.13}$$

Remark. *1st Matching Condition* of torsion of ∇ with L, or MC1, resembles the compatibility condition between a symplectic form ω and L, namely, $\omega(LX, Y) = \omega(X, LY)$. For this reason, we also refer to MC1 as "torsion-compatibility." *2nd Matching Condition* of torsion of ∇ with L, or MC2, was referred to as "torsion-coupling" in [7], where a comprehensive study was conducted on how such coupling interacts with Codazzi coupling of (∇, g) in almost (para-)Hermtian manifolds $(\mathcal{M}, g, L)$.

Proposition 3 *The Nijenhuis tensor N_L vanishes if*

(a) MC1 (torsion-compatibility) holds for both ∇ and ∇^L; or
(b) MC2 (torsion-coupling) holds for either ∇ or ∇^L.

Proof With respect to statement (a), substituting the following torsion-compatibility conditions

$$T^{\nabla}(LX, Y) + T^{\nabla}(X, LY) = 0$$
$$L^2\, T^{\nabla^L}(X, Y) + T^{\nabla^L}(LX, LY) = 0$$

into Eq. (3.9) leads to $N_L = 0$. With respect to statement (b), substituting Eq. (3.13), the torsion-coupling condition, into Eq. (3.8) leads to $N_L = 0$.

Proposition 4 *With respect to the following statements*

(i) $N_L = 0$*;*
(ii) (∇, L) *satisfies MC1, i.e., Eq. (3.11);*
(iii) (∇^L, L) *satisfies MC1;*
(iv) (∇, L) *satisfies MC2, i.e., Eq. (3.12) or Eq. (3.13);*
(v) (∇^L, L) *satisfies MC2;*

we have

(a) (iv) and (v) are equivalent;
(b) (iv) or (v) implies (i);
(c) any two of (i), (ii), (iii) implies the third;
(d) either (iv) or (v) together with (ii) or (iii) imply that

$$T^{\nabla} = T^{\nabla^L} = 0.$$

Proof That (∇, L) satisfies MC2 means

$$T^{\nabla^L}(LX, Y) = L^{-1}T^{\nabla}(L(LX), Y) = L^{-1}(T^{\nabla}(L^2X, Y)) = L(T^{\nabla}(X, Y)).$$

So, that (∇, L) satisfies MC2 is equivalent to that (∇^L, L) satisfies MC2. Hence (iv) and (v) are equivalent statements. And each implies $N_L = 0$.

That (ii) plus (iii) imply (i) is what statement (a) of Proposition 3 indicates. To show (i) and (ii) leads to (iii), we only need to mention that having assumed (ii), the formula for the Nijenhuis tensor reduces to

$$N_L(X, Y) = L^2 T^{\nabla^L}(X, Y) + T^{\nabla^L}(LX, LY).$$

Further assuming (i) leads to (iii).

Finally, we show that (iv) will lead to $T^\nabla(LX, Y) = T^\nabla(X, LY)$:

$$T^\nabla(LX, Y) = LT^{\nabla^L}(X, Y) = -LT^{\nabla^L}(Y, X) = -T^\nabla(LY, X) = T^\nabla(X, LY).$$

And likewise, $T^{\nabla^L}(LX, Y) = T^{\nabla^L}(X, LY)$. Therefore, further imposing (ii) or (iii) forces $T^\nabla = T^{\nabla^L} = 0$.

Remark. In the special case $\nabla = \nabla^L$, then ∇ is a (para-)complex connection $\nabla L = 0$. In this case, the last two terms of the righthand side of Eq. (3.12) vanish. So $T^\nabla(LX, Y) = LT^\nabla(X, Y)$. A complex connection that is metric and satisfies Eq. (3.12) (torsion-coupling condition) is known as a *second canonical connection* for Hermitian manifolds. It is also called the *Chern connection*. On the other hand, a complex connection that is metric and satisfies Eq. (3.11) (torsion-compatibility condition), $T^\nabla(LX, Y) + T^\nabla(X, LY) = 0$, is known as a *first canonical connection* for Hermitian manifolds. So our definitions of MC1 (torsion-compatibility) and MC2 (torsion-coupling) generalize the appropriate definitions of first and second canonical connections with torsion, to non-(para-)complex connections with torsion. Part (d) of Proposition 4 further claims that MC1 and MC2 are genuinely different types of coupling, if ∇ and ∇^L must carry non-zero torsions.

3.3 Torsion of ∇ Under (Para-)complexification

3.3.1 Splitting of $T\mathcal{M} \otimes \mathbb{L}$ by L

For a smooth manifold $\mathcal{M}$, an endomorphism of the tangent bundle $T\mathcal{M}$ induces a smooth map L of the tangent bundle. By definition, L is called an *almost complex structure* if $L^2 = -\mathrm{id}$, or an *almost para-complex structure* if $L^2 = \mathrm{id}$ and the multiplicities of the eigenvalues ± 1 are equal. We have used J and K to denote almost complex structures and almost para-complex structures, respectively, and use L when these two structures can be treated in a unified way. It is clear from our definition that such structures exist only when $\mathcal{M}$ is of even dimension.

Denote eigenvalues of L as $\pm\alpha$, where $\alpha = 1$ for $L = K$ and $\alpha = i$ for $L = J$, depending on the nature of L. We have

$$L^4 = \mathrm{id}, \qquad \alpha^4 = 1, \qquad \alpha^2 L^2 = \mathrm{id}.$$

Following the standard procedure, we (para-)complexify $T\mathcal{M}$ by tensoring with complex field $\mathbb{C}$ or the para-complex (a.k.a. split-complex) field $\mathbb{D}$, and use $T^{\mathbb{L}}\mathcal{M}$ to denote the resulting $T\mathcal{M} \otimes \mathbb{C} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M} \otimes \mathbb{C}$ or $T\mathcal{M} \otimes \mathbb{D} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M} \otimes \mathbb{D}$, depending on the type of L. In analogy with standard notation in the complex case, let $T_p^+\mathcal{M}$ and $T_p^-\mathcal{M}$ be the eigenbundles of L corresponding to the eigenvalues $\pm\alpha$, i.e., at each point $p \in \mathcal{M}$, the fiber is defined by

$$T_p^+\mathcal{M} := \{Z \in T_p^{\mathbb{L}}\mathcal{M} : L_p(Z) = \alpha Z\} ,$$
$$T_p^-\mathcal{M} := \{Z \in T_p^{\mathbb{L}}\mathcal{M} : L_p(Z) = -\alpha Z\} .$$

with

$$T^{\mathbb{L}}\mathcal{M} \equiv T\mathcal{M} \otimes \mathbb{L} = \bigcup_{p\in\mathcal{M}} T_p^+\mathcal{M} \oplus T_p^-\mathcal{M} = T^+\mathcal{M} \oplus T^-\mathcal{M}.$$

We will refer to vectors to be of "type $+$" or $(1, 0)$, or "type $-$" or $(0, 1)$, if they take values in $T_p^+\mathcal{M}$ or $T_p^-\mathcal{M}$, respectively. Moreover, define π^+ and π^- to be the projections of a (para-)complexified vector field to $T^+\mathcal{M} = \bigcup_p T_p^+\mathcal{M}$ and $T^-\mathcal{M} = \bigcup_p T_p^-\mathcal{M}$, respectively:

$$\pi^+(Z) = \frac{1}{2}(Z + \alpha^{-1}LZ), \qquad \pi^-(Z) = \frac{1}{2}(Z - \alpha^{-1}LZ).$$

As subbundles of the (para-)complexified tangent bundle $T^{\mathbb{L}}\mathcal{M}$, $T^+\mathcal{M}$ and $T^-\mathcal{M}$ are distributions. A distribution is called a foliation if it is closed under the Lie bracket $[\cdot, \cdot]$.

Let $T_p^{(n,m)}\mathcal{M}$ denotes the tensor product of n-copies of $T_p^+\mathcal{M} \equiv T_p^{(1,0)}\mathcal{M}$ and m-copies of $T_p^-\mathcal{M} \equiv T_p^{(0,1)}\mathcal{M}$:

$$T_p^{(n,m)}\mathcal{M} = \underbrace{(T_p^+\mathcal{M} \times \cdots \times T_p^+\mathcal{M})}_{n \text{ times}} \times \underbrace{(T_p^-\mathcal{M} \times \cdots \times T_p^-\mathcal{M})}_{m \text{ times}}.$$

In particular, $T_p^{(2,0)}\mathcal{M}$ will supply two vectors fields both drawn from $T_p^+\mathcal{M}$, $T_p^{(0,2)}\mathcal{M}$ will supply two vectors fields both drawn from $T_p^-\mathcal{M}$, and $T_p^{(1,1)}\mathcal{M}$ will supply two vectors fields drawn respectively from $T_p^+\mathcal{M}$ and $T_p^-\mathcal{M}$. That is, the duplet vector fields (X, Y) take in value, respectively, from $(+, +)$, from $(+, -)$ or $(-, +)$, or from $(-, -)$, which are shorthand notations for $T_p^{(2,0)}\mathcal{M}$, $T_p^{(1,1)}\mathcal{M}$, or $T_p^{(0,2)}\mathcal{M}$.

It is important to understand what (para-)complexification accomplishes. Take $L = J$, for example. Because $J^2 = -\mathrm{id}$, the J operator on a $dim(\mathcal{M}) = 2n$ dimensional vector space (i.e., J maps a real-valued $2n$-vector to a real-valued $2n$-vector) does not have any eigenvector over the real field $\mathbb{R}$: that is, there is no $X \in T_p\mathcal{M}$ such that $JX = \alpha X$ where $\alpha = \pm i$ are the two eigenvalues of the $2n \times 2n$ real-valued operator J.

Given a vector $X \in T_p\mathcal{M}$ on the real even-dimensional manifold, (para-)complexification amounts to supplying it with an "imaginary" component, to yield an element $Z \in T_p^{\mathbb{L}}\mathcal{M} \equiv T_p\mathcal{M} \otimes \mathbb{L}$. Even better, we can turn X into either X^+ or X^-, where

$$X^{+} = X + \alpha^{-1} LX \in T_p^{+}\mathcal{M}, \quad X^{-} = X - \alpha^{-1} LX \in T_p^{-}\mathcal{M}.$$

The (1,0) or “plus”-eigenspace $T_p^{+}\mathcal{M}$ and the (0,1) or “minus”-eigenspace $T_p^{-}\mathcal{M}$ are formed, respectively, by such X^{+} and X^{-} having been supplemented with the imaginary parts $\alpha^{-1}LX$ or $-\alpha^{-1}LX$:

$$T_p^{+}\mathcal{M} := \{X^{+} = X + \alpha^{-1}LX : X \in T_p\mathcal{M}\},$$
$$T_p^{-}\mathcal{M} := \{X^{-} = X - \alpha^{-1}LX : X \in T_p\mathcal{M}\}.$$

Note that for any (para-)complexified vector field $Z \in T^{\mathbb{L}}\mathcal{M}$, the vector field $Z^{+} = Z + \alpha^{-1}LZ$ is always in $T_p^{+}\mathcal{M}$ and the vector field $Z^{-} = Z - \alpha^{-1}LZ$ is always in $T_p^{-}\mathcal{M}$. This can be easily shown:

$$LZ^{+} = \alpha Z^{+}, \quad LZ^{-} = -\alpha Z^{-}$$

because

$$L(Z + \alpha^{-1}LZ) \equiv \alpha(Z + \alpha^{-1}LZ), \qquad L(Z - \alpha^{-1}LZ) \equiv -\alpha(Z - \alpha^{-1}LZ).$$

Thus, any vector $Z \in T_p^{\mathbb{L}}\mathcal{M}$ can be readily decomposed into $T_p^{+}\mathcal{M}$ and $T_p^{-}\mathcal{M}$. Without loss of generality, $T_p^{+}\mathcal{M}$ is formed by vectors of the form $X^{+} = X + \alpha^{-1}LX$, and $T_p^{-}\mathcal{M}$ of vectors of the form $X^{-} = X - \alpha^{-1}LX$, when X exhausts the real vector field as sections of $T_p\mathcal{M}$.

3.3.2 *(Para-)holomorphicity of* ∇

The splitting of $T\mathcal{M} \otimes \mathbb{L}$ by L into direct sum of $T_p^{+}\mathcal{M}$ and $T_p^{-}\mathcal{M}$ subbundles gives rise to questions of whether/how ∇ respects such splitting.

3.3.2.1 (Para)-Dolbeault Operator $\bar{\partial}$

The (para-)Dolbeault operator $\bar{\partial}$ on $\mathcal{M}$ for a given L is defined as [8]

$$\bar{\partial}_X Y = \frac{1}{4}\left([X,Y] - L^{-1}[LX,Y] + L^{-1}[X,LY] - L^{2}[LX,LY]\right) \tag{3.14}$$

for any vector fields X and Y in $\Xi(T\mathcal{M})$. It can be checked that $\bar{\partial}_X Y$ is tensorial in X, such that $\bar{\partial}_{fX} Y = f\left(\bar{\partial}_X Y\right)$, and is a derivation in Y, such that the Leibniz rule is satisfied. It can be easily verified that

$$\bar{\partial}_{LX} Y = -L(\bar{\partial}_X Y), \qquad \bar{\partial}_X(LY) = L(\bar{\partial}_X Y).$$

Using definition Eq. (3.7) for N_L, we can rewrite the definition (3.14) of $\bar{\partial}$ as

$$\bar{\partial}_X Y = \frac{1}{2}\left([X,Y] - L^{-1}[LX,Y] + \frac{1}{2}L^2 N_L(X,Y)\right). \tag{3.15}$$

When we extend $\bar{\partial}$ to (para-)complexified tangent bundles, then we will find that

$$\begin{aligned}\bar{\partial}_{X^+}Y^+ &= \bar{\partial}_{X^-}Y^- = 0;\\ \bar{\partial}_{X^-}Y^+ &= \pi^+[X^-,Y^+];\\ \bar{\partial}_{X^+}Y^- &= \pi^-[X^+,Y^-].\end{aligned}$$

Stated otherwise, $\bar{\partial}Y^+$ is a vector-valued 1-form, of type $+$ as a vector and type $-$ as a 1-form, and conversely in types for $\bar{\partial}Y^-$.

It is important to note that this operator $\bar{\partial}$ is conventionally deployed only when the almost complex structure J is integrable (i.e., becomes a complex structure), whence $T^+\mathcal{M}$ becomes a holomorphic tangent bundle (with holomorphic coordinate z on the base manifold) and $T^-\mathcal{M}$ an anti-holomorphic tangent bundle (with anti-holomorphic coordinates $\bar{z}$ on the base manifold). It is also known as the *intrinsic Cauchy-Riemann operator* of J, since it only depends on the almost-complex structure J. $\bar{\partial}$ usually operates on $T^+\mathcal{M}$ only, and induces the differentiation of tangent vector fields in $T^+\mathcal{M}$ with respect to the anti-holomorphic coordinates $\frac{\partial}{\partial\bar{z}}$. The operation of $\bar{\partial}$ on $T^-\mathcal{M}$ with respects to $\frac{\partial}{\partial\bar{z}}$ will be identically zero.

In our current usage of $\bar{\partial}$, we do not assume integrability of L, so it makes sense to consider $\bar{\partial}$ on the whole (para-)complexified tangent bundle $T^{\mathbb{L}}\mathcal{M}$. In this more general usage, $\bar{\partial}$ is said to define a *pseudo-(para-)holomorphic* structure independent of whether L is integrable or not. For convenience and brevity, we drop the prefix "pseudo."

3.3.2.2 (Para-)holomorphic Connections

Given a connection ∇ operating on $T^{\mathbb{L}}\mathcal{M}$, we can ask the question whether ∇ as a covariant derivative is compatible with $\bar{\partial}$. To understand this we may define an alternative operator $\bar{\partial}^{\nabla}$, see [8]

$$\bar{\partial}^{\nabla}_X Y = \frac{1}{2}\left(\nabla_X Y - \nabla_{LX}(L^{-1}Y)\right) \tag{3.16}$$

and extend the vector fields X and Y to (para-)complexified ones in $T^{\mathbb{L}}\mathcal{M}$. It satisfies

$$\bar{\partial}^{\nabla}_X(LY) = -\bar{\partial}^{\nabla}_{LX}(Y).$$

Clearly,

$$\bar{\partial}^{\nabla}_{X^+} Y^+ = \bar{\partial}^{\nabla}_{X^-} Y^- = 0;$$
$$\bar{\partial}^{\nabla}_{X^-} Y^+ = \nabla_{X^-} Y^+;$$
$$\bar{\partial}^{\nabla}_{X^+} Y^- = \nabla_{X^+} Y^-.$$

In other words, operating on $Y^+ \in T^+\mathcal{M}$, $\bar{\partial}^{\nabla}$ takes the (0,1)-part of the vector-valued 1-form ∇Y^+ (and conversely the (1,0)-part when operating on $Y^- \in T^-\mathcal{M}$).

Again, we do not assumed integrability of L, so the same caveat applies to $\bar{\partial}^{\nabla}$ as for the operator $\bar{\partial}$ defined above—$\bar{\partial}^{\nabla}$ is defined on the full (para-)complexified tangent bundle $T^{\mathbb{L}}\mathcal{M}$, and therefore a restriction of it to $T^-\mathcal{M}$ would actually be denoted by ∂^{∇} if acting on $T^-\mathcal{M}$ as a (para-)holomorphic bundle.

On a (para-)holomorphic vector bundle, a connection is conventionally called *(para-)holomorphic* if the two Dolbeault operators $\bar{\partial}$ and $\bar{\partial}^{\nabla}$ coincide. We extend this notion to arbitrary connections on $T^{\mathbb{L}}\mathcal{M} = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}$ where ∇ does not necessarily preserve $T^{(1,0)}\mathcal{M}$ or $T^{(0,1)}\mathcal{M}$.

Definition 2 A connection ∇ is called (para-)holomorphic if $\bar{\partial}^{\nabla}_X Y = \bar{\partial}_X Y$ for any vector fields X and Y.

Using (3.14) and (3.16), we can also derive a necessary and sufficient condition for (para-)holomorphicity in terms of N_L, T^{∇}, and T^{∇^L}. In fact the following were proven in [7]:

Lemma 2 *Given an arbitrary pair* (∇, L)*, the following statements are equivalent*

(i) ∇ *is (para-)holomorphic;*
(ii) ∇^L *is (para-)holomorphic;*
(iii) *the following holds*

$$\frac{1}{2} N_L(X, Y) = L^2\, T^{\nabla^L}(X, Y) - L T^{\nabla}(LX, Y). \tag{3.17}$$

Two connections ∇^1 and ∇^2 are said to be $\bar{\partial}$-balanced when $\bar{\partial}^{\nabla^1}_X Y = \bar{\partial}^{\nabla^2}_X Y$ holds. This Lemma implies that a sufficient condition for (∇, ∇^L) to be $\bar{\partial}$-balanced is that either of them is (para-)holomorphic. (Para-)holomorphicity of ∇ (and ∇^L) is stronger than (∇, ∇^L) being $\bar{\partial}$-balanced. These notions were investigated in [7].

3.3.3 (Para-)complexifying N_L and T^{∇}

3.3.3.1 Decomposition of N_L

Both $T^{(1,0)}\mathcal{M}$ and $T^{(0,1)}\mathcal{M}$ are foliations if and only if L is integrable, i.e., the integrability condition $N_L = 0$ is satisfied.

We recall that in the (para-)complex case, the $(1, 1)$-component of the Nijenhuis tensor vanishes identically. Heuristically, this follows from the fact that the Nijenhuis tensor is (apart from a multiplicative constant) the (0, 1)-part of the Lie bracket of two (1, 0)-vector fields $N(Z_1, Z_2) = \pi^-[\pi^+ Z_1, \pi^+ Z_2]$. Therefore, we do not expect there to be any (1, 1)-component. To see this directly, let X and Y be real vector fields, from which the (1,0)-vector X^+ and (0,1)-vector Y^- can be constructed. We have

$$\begin{aligned} N^{(1,1)} &= N(X^+, Y^-) \\ &= -L^2[X^+, Y^-] + L[LX^+, Y^-] + L[X^+, LY^-] - [LX^+, LY^-] \\ &= -L^2[X^+, Y^-] + \alpha L[X^+, Y^-] - \alpha L[X^+, Y^-] + \alpha^2[X^+, Y^-] \\ &= 0 \end{aligned}$$

where the last step invoked $\alpha^2 L^2 = \mathrm{id}$.

The (2, 0)-part of the Nijenhuis tensor, namely taking inputs from $T_p^{(2,0)}\mathcal{M} = T_p^+\mathcal{M} \times T_p^+\mathcal{M}$ or $(+, +)$, can be calculated as the following

$$\begin{aligned} N^{(2,0)} &= N(X^+, Y^+) \\ &= -L^2[X^+, Y^+] + L[LX^+, Y^+] + L[X^+, LY^+] - [LX^+, LY^+] \\ &= -L^2[X^+, Y^+] + \alpha L[X^+, Y^+] + \alpha L[X^+, Y^+] - \alpha^2[X^+, Y^+] \\ &= -2L^2 \left([X^+, Y^+] - \alpha^{-1} L[X^+, Y^+]\right) \\ &= -4\, L^2\, \pi^-([X^+, Y^+]) \\ &= -4\alpha^2\, \pi^-([X^+, Y^+]). \end{aligned}$$

And similarly

$$N^{(0,2)} = N(X^-, Y^-) = -4\alpha^2\, \pi^+([X^-, Y^-]).$$

On the other hand,

$$\begin{aligned} N^{(2,0)} &= N(X^+, Y^+) = N(X + \alpha^{-1}LX, Y + \alpha^{-1}LY) \\ &= N(X, Y) + \alpha^{-1}N(LX, Y) + \alpha^{-1}N(X, LY) + \alpha^{-2}N(LX, LY) \\ &= N(X, Y) - \alpha^{-1}LN(X, Y) - \alpha^{-1}LN(X, Y) + \alpha^{-2}L^2\, N(X, Y) \\ &= 2\, N(X, Y) - 2\alpha^{-1}LN(X, Y) = 4\pi^-(N(X, Y)), \end{aligned}$$

and similarly

$$N^{(0,2)} = 4\pi^+(N(X, Y)).$$

The above derivation shows that N_L, when defined over (para-)complexified vector fields, decompose into three parts:

(i) $N^{(1,1)}$, which is identically zero;
(ii) $N^{(2,0)}$, which may not be zero;
(iii) $N^{(0,2)} \equiv \overline{N^{(2,0)}}$.

Therefore, to show that the (para-)complexified Nijenhuis tensor vanishes, it is sufficient to show that its $(2, 0)$-part does. The nature of $N^{(2,0)}$ will be investigated in Sect. 3.3.5.1.

3.3.3.2 (Para-)complexifying MC1 and MC2

The torsion tensor T^∇ of a connection ∇ on $\mathcal{M}$ is a 1,2-tensor, that is, it takes in two (real-valued) vector fields X, Y and outputs one (real-valued) vector-field denoted by $T^\nabla(X, Y)$. When the vector fields are (para-)complexified, we can treat T^∇ as mapping two (para-)complexified vector fields to one (para-)complexified vector field as an output. Naturally, we can investigate its various components, taking inputs in, $T_p^+\mathcal{M} \times T_p^+\mathcal{M}$ or $(+, +)$ called "(2,0)-part", $T_p^+\mathcal{M} \times T_p^-\mathcal{M}$ or $(+, -)$ called "(1,1)-part", $T_p^-\mathcal{M} \times T_p^-\mathcal{M}$ or $(-, -)$ called "(0,2)-part".

Because torsion can be decomposed into (2,0)-, (1,1)-, or (0,2)-part, we can also investigate the 1st Matching Condition (MC1) and 2nd Matching Condition (MC2) between T^∇ and L when torsion T^∇ breaks down in terms of these separate components. We refer to this as "(para-)complexifying" torsion-coupling and torsion-compatibility conditions. In doing so, we hope to obtain a deeper appreciation of whether enforcing these conditions separately on each (para-) holomorphic/anti-(para-)holomorphic components affects integrability of L and/or (para-)holomorphicity of ∇.

With respect to the "torsion-coupling" of (∇, L) or MC2, we will show that the $(2, 0) + (0, 2)$-part is equivalent to the vanishing of the Nijenhuis tensor, while the $(1, 1)$-part is equivalent to the holomorphicity of ∇. Since torsion-coupling of ∇ implies torsion-coupling of ∇^L, enforcing $(1, 1)$-part of torsion-coupling also implies that ∇^L is holomorphic (just as ∇ is). Furthermore, the $(1, 1)$-part of torsion-coupling implies that the $(1, 1)$-component of the (para-)complexified torsion is 0, both for ∇ and ∇^L.

With respect to the "torsion-compatibility" of (∇, L) or MC1, we will show that the (para-)complexified version of this condition Eq. (3.11) implies that the $(2, 0) + (0, 2)$-part of the torsion must vanish; the torsion carried by ∇, if any, is entirely of $(1, 1)$ type. Because the (2,0)- and (0,2)-part are equivalent, which is meant by (2,0) + (0,2)-part, we will use the phrase (2,0)-part or (2,0)-component from now on.

The next two subsections give the details of these assertions.

3.3.4 Torsion-Compatibility (MC1)

3.3.4.1 (1,1)-Part of Torsion-Compatibility

It is easy to verify that

$$T^{\nabla}(LX^{+}, Y^{-}) + T^{\nabla}(X^{+}, LY^{-}) = \alpha T^{\nabla}(X^{+}, Y^{-}) - \alpha T^{\nabla}(X^{+}, Y^{-}) \equiv 0$$

for any real vector fields X, Y. This is to say, the (1,1)-part of the torsion-compatibility condition is *always* satisfied.

3.3.4.2 (2,0)-Part of Torsion-Compatibility

Let us now require torsion-compatibility to hold for the (2,0)-part:

$$T^{\nabla}(LX^{+}, Y^{+}) + T^{\nabla}(X^{+}, LY^{+}) = 0.$$

Writing this out explicitly,

$$\begin{aligned}
0 &= 2\alpha T^{\nabla}(X^{+}, Y^{+}) = 2\alpha(T^{\nabla}(X + \alpha^{-1}LX, Y + \alpha^{-1}LY)) \\
&= 2\alpha(T^{\nabla}(X, Y) + \alpha^{-2}T^{\nabla}(LX, LY) + \alpha^{-1}T^{\nabla}(LX, Y) + \alpha^{-1}T^{\nabla}(X, LY)) \\
&= 2\,T^{\nabla}(LX, Y) + 2\,T^{\nabla}(X, LY) + \alpha^{-1}\left(2\,L^{2}\,T^{\nabla}(X, Y) + 2\,T^{\nabla}(LX, LY)\right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
T^{\nabla}(LX, Y) + T^{\nabla}(X, LY) &= 0, \\
L^{2}T^{\nabla}(X, Y) + T^{\nabla}(LX, LY) &= 0.
\end{aligned}$$

The first equation is the torsion-compatibility we started with. The second equation is identical to the first one, after setting LX for X in the first equation. So no new conditions are obtained after (para-)complexifying torsion-compatibility. Summarizing the situation for both (1,1)- and (2,0)-components of the torsion tensor, (para-)complexifying torsion-compatibility (MC1) does not yield any new information beyond saying that the MC1 condition amounts to the (2,0)-component of torsion being zero.

3.3.4.3 Para-Complexifying Codazzi Coupling

Codazzi coupling of ∇ and L leads to, according to Proposition 1,

$$T^{\nabla}(X, Y) = T^{\nabla^{L}}(X, Y).$$

(Para-)complexifying the above, whether taking (2,0)- or (1,1)-component, leads to

$$\begin{aligned}
T^{\nabla}(LX, Y) + T^{\nabla}(X, LY) &= T^{\nabla^{L}}(LX, Y) + T^{\nabla^{L}}(X, LY), \\
L^{2}\,T^{\nabla}(X, Y) + T^{\nabla}(LX, LY) &= L^{2}T^{\nabla^{L}}(X, Y) + T^{\nabla^{L}}(LX, LY).
\end{aligned}$$

Substituting $X \to LX$ in the first equation leads to the second equation. So (para-)complexifying Codazzi coupling leads to a relaxation of the relation $T^{\nabla}(X, Y) = T^{\nabla^L}(X, Y)$.

Codazzi coupling, along with MC1 applied to either ∇ or ∇^L, will lead to MC1 satisfied for both ∇ and ∇^L and hence, the integrability of L.

3.3.5 Torsion-Coupling (MC2)

In contrast to MC1, the situation is different when (para-)complexifying torsion-coupling (MC2)—after complexifying, the (2,0)- and (1,1)-part reveal independent information, as we will show now.

3.3.5.1 (2,0)-Part of Torsion-Coupling: Integrability

From Sect. 3.3.3.1, we see that the only non-vanishing part of the (para-)complexfied N_L is $N^{(2,0)}$ (and $N^{(0,2)}$). In this subsection, we relate $N^{(0,2)}$ to MC2 or torsion-coupling.

Proposition 5 *The (2,0)-part $N^{(2,0)}$ of N_L vanishes if and only if the (2,0)-part of torsion-coupling is enforced. That is,*

$$N^{(2,0)} := N_L(X^+, Y^+) = 0$$

if and only if

$$LT^{\nabla^L}(X^+, Y^+) = T^{\nabla}(LX^+, Y^+).$$

Proof Instead of enforcing the full torsion-coupling condition, we only assume that the $(2, 0)$-part of the torsion is coupled to L, and show that it is equivalent to the vanishing of the Nijenhuis tensor.

$$\begin{aligned}
&\frac{1}{2} N_L(X^+, Y^+) \\
&= \frac{1}{2}\Big(T^{\nabla^L}(L^2X^+, Y^+) + T^{\nabla^L}(LX^+, LY^+) - LT^{\nabla}(LX^+, Y^+) - LT^{\nabla}(X^+, LY^+)\Big) \\
&= \frac{1}{2}\Big(L^2\, T^{\nabla^L}(X^+, Y^+) + \alpha^2\, T^{\nabla^L}(X^+, Y^+) - 2\alpha L\, T^{\nabla}(X^+, Y^+)\Big) \\
&= L^2\, T^{\nabla^L}(X^+, Y^+) - \alpha LT^{\nabla}(X^+, Y^+) \\
&= L\Big(LT^{\nabla^L}(X^+, Y^+) - T^{\nabla}(LX^+, Y^+)\Big).
\end{aligned}$$

The expression in the parenthesis is the expression of torsion-coupling applied to (2,0)-part.

Let us now writing out explicitly the (2,0)-part of torsion-coupling

$$L\left(T^{\nabla^L}(X+\alpha^{-1}LX, Y+\alpha^{-1}LY)\right) = \alpha T^{\nabla}(X+\alpha^{-1}LX, Y+\alpha^{-1}LY)$$

or explicitly,

$$\begin{aligned} &L\left(T^{\nabla^L}(X,Y)+\alpha^{-2}T^{\nabla^L}(LX,LY)+\alpha^{-1}(T^{\nabla^L}(LX,Y)+T^{\nabla^L}(X,LY))\right) \\ &= \alpha\left(T^{\nabla}(X,Y)+\alpha^{-2}T^{\nabla}(LX,LY)\right)+T^{\nabla}(LX,Y)+T^{\nabla}(X,LY), \end{aligned}$$

from which we obtain two equations (separating real and imaginary components):

$$\begin{aligned} L^2\, T^{\nabla^L}(X,Y)+T^{\nabla^L}(LX,LY) &= L\left(T^{\nabla}(X,LY)+T^{\nabla}(LX,Y)\right), \\ L\left(T^{\nabla^L}(LX,Y)+T^{\nabla^L}(X,LY)\right) &= L^2\, T^{\nabla}(X,Y)+T^{\nabla}(LX,LY). \end{aligned}$$

The above two equations are equivalent to each other, since substituting LX for X in the first yields the second. Furthermore they exactly (!) express the condition $N_L = 0$. This shows that the (2,0)-part of the torsion-coupling condition is exactly $N_L = 0$. Since $N^{(1,1)} \equiv 0$, $N_L = 0$ is equivalent to $N^{(2,0)} = 0$.

3.3.5.2 (1, 1)-Part of the Torsion-Coupling: (Para-)holomorphicity

Let us now only impose torsion-coupling condition to the (1,1)-part, that is, when inputs are taken from $T_p^+\mathcal{M} \times T_p^-\mathcal{M}$ or $(+,-)$:

$$LT^{\nabla}(X+\alpha^{-1}LX, Y-\alpha^{-1}LY) = \alpha T^{\nabla^L}(X+\alpha^{-1}LX, Y-\alpha^{-1}LY).$$

Writing out explicitly,

$$\begin{aligned} &L\left(T^{\nabla}(X,Y)-\alpha^{-2}T^{\nabla}(LX,LY)+\alpha^{-1}T^{\nabla}(LX,Y)-\alpha^{-1}T^{\nabla}(X,LY)\right) \\ &= \alpha\left(T^{\nabla^L}(X,Y)-\alpha^{-2}T^{\nabla^L}(LX,LY)+\alpha^{-1}T^{\nabla^L}(LX,Y)-\alpha^{-1}T^{\nabla^L}(X,LY)\right) \end{aligned}$$

from which we obtain two equations (separating real and imaginary components):

$$\begin{aligned} L\left(T^{\nabla}(X,Y)-L^2\, T^{\nabla}(LX,LY)\right) &= T^{\nabla^L}(LX,Y)-T^{\nabla^L}(X,LY), \\ L\left(T^{\nabla}(LX,Y)-T^{\nabla}(X,LY)\right) &= L^2 T^{\nabla^L}(X,Y)-T^{\nabla^L}(LX,LY). \end{aligned}$$

The above two equations are equivalent to each other—we can obtain the second from subsituting $X \to LX$ in the first. Rearranging the second:

$$T^{\nabla^L}(LX,LY)-LT^{\nabla}(X,LY) = L^2T^{\nabla^L}(X,Y)-LT^{\nabla}(LX,Y). \tag{3.18}$$

We call the above Eq. (3.18) "Condition H". It is the $(1,1)$-part of the (para)complexified torsion-coupling condition, which is a slight relaxation to the standard torsion-coupling condition. If the lefthand side (and the righthand side) equals 0, that is the standard torsion-coupling condition Eq. (3.13).

Proposition 6 *Condition H or Eq. (3.18), which expresses the (1,1)-part of torsion-coupling, is satisfied if and only if* ∇ *is (para-)holomorphic.*

Proof Recall from Proposition 2

$$N_L(X,Y) = \underbrace{\Big(T^{\nabla^L}(L^2X,Y) - LT^{\nabla}(LX,Y)\Big)}_{A} + \underbrace{\Big(T^{\nabla^L}(LX,LY) - LT^{\nabla}(X,LY)\Big)}_{B}.$$

Eq. (3.18) amounts to requiring the underbracketed "A" and "B" are equal. Hence, it is equivalent to

$$\frac{1}{2}N_L(X,Y) = L^2T^{\nabla^L}(X,Y) - LT^{\nabla}(LX,Y).$$

By Lemma 2, this is equivalent to ∇ being (para-)holomorphic.

3.4 Summary and Discussions

Affine connections on almost Hermitian manifold are investigated with a typical assumption that these connections ∇ are complex: $\nabla J = 0$. When torsions are to be considered, it is widely known that two types of torsion exist that nevertheless can lead to integrable complex manifold (vanishing of N_J):

$$T(JX,JY) = T(X,Y), \qquad \text{torsion is (1,1)-type,} \tag{3.19}$$

$$T(JX,Y) = JT(X,Y), \qquad \text{torsion is (2,0)-type.} \tag{3.20}$$

If furthermore ∇ is a metric connection, $\nabla g = 0$, then the (1,1)-type is called the *first canonical connection* and the (2.0)-type the *second canonical connection*.

The second canonical connection is also known as Chern connection. It is a holomorphic connection, and is uniquely determined. Reference [11] classified 16 classes of almost Hermitian structures based on its Levi-Civita connection. In general, Levi-Civita connection and Chern connection are not the same on a Hermitian manifold; they are one and the same if the Hermitian manifold is Kähler, where the torsion of the Chern connection vanishes.

Our paper extends these considerations to arbitrary connections that are in general not parallel with respect to L, $\nabla L \neq 0$. The approach is through dualistic approach, by considering ∇^L, the conjugate connection with respect to L. The unique L-conjugate connection ∇^L has the property that $\frac{1}{2}(\nabla + \nabla^L)L = 0$. Either of these connections may carry torsion. So our paper characterizes the torsion tensors T^{∇} and T^{∇^L} that will lead to integrable (para-)complex structure. The case of $\nabla L = 0$ is recovered as the special case where $\nabla = \nabla^L$.

A Hermitian connection ∇ is one that preserves the Hermitian form h, $\nabla h = 0$, defined by $h(X, Y) = g(X, Y) + \sqrt{-1}\omega(X, Y)$. Equivalently, a Hermitian connection is one which is metric ($\nabla g = 0$) and complex ($\nabla J = 0$). In general, Hermitian connections carry non-zero torsion. In fact, there is a line of Hermitian connections called the *Gauduchon connections*, all with differerent torsions. This line passes through the first canonical connection and the second canonical connection which satisfies, respectively, Eq. (3.19) and Eq. (3.20). But there are infinitely many other Hermitian connections that do not fall on the Gauduchon line.

Our investigations generalize the first and second canonical connections by removing the assumption of their being (para-)complex connections $\nabla L \neq 0$. Specifically, MC2 generalizes Eq. (3.20) to involve both ∇, ∇^L, while MC1 is to be enforced separately on ∇ and ∇^L. The set of torsion-compatible connections (satisfying MC1) intersects the line of Gauduchon connections at the first canonical connection. This intersection is unique (and transversal) because for all other Gauduchon connections, the torsion is not $(1, 1)$. The set of torsion-coupled connections (satisfying MC2) intersects the line of Gauduchon connections at the Chern connection (the second canonical connection). This intersection is also unique and transversal because the Chern connection is the unique holomorphic Hermitian (complex and metric) connection.

MC1 condition expressed a constraint about the torsion of ∇, namely, the (2,0)- and (0,2)-component of T^{∇} must vanish. However, this constraint can be *separately* enforced upon ∇ and ∇^L. Only when both T^{∇} and T^{∇^*} satisfy MC1 would an integrable L result. In the presence of Codazzi coupling, which stipulates $T^{\nabla} = T^{\nabla^L}$, (∇, L) satisfies MC1 if and only if (∇^L, L) satisfies MC1. Thus MC1 articulates a generalization of the first canonical connection.

MC2 condition, on the other hand, expresses a "balance" of T^{∇} and T^{∇^L}, such that it involves the torsions of ∇ and ∇^L in an intertwined fashion. (∇, L) satisfies MC2 if and only if (∇^L, L) satisfies MC2. It is in this scenario that ∇ and ∇^L both become (para-)holomorphic connections. MC2 articulates a generalization of the second canonical connection.

$$\begin{cases} (2,0)\text{-part of MC2} \Leftrightarrow N^{(2,0)} = 0 \Leftrightarrow N_L(X, Y) = 0, \quad \text{since } N^{(1,1)} \equiv 0 \\ (1,1)\text{-part of MC2} \Leftrightarrow \quad \text{Cond. H} \quad \Leftrightarrow \quad \text{(para-)holomorphicity of } \nabla \end{cases}$$

MC1 and MC2 conditions are independent of each other. When torsion-coupling and torsion-compatibility are both imposed, then the connections must be torsion-free $T^{\nabla} = T^{\nabla^L} = 0$. These are torsion-free holomorphic connections yet generally not (para-)complex.

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