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(Para-)Holomorphic and Conjugate Connections on (Para-)Hermitian and (Para-)Kähler Manifolds

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Abstract. We investigate how an affine connection ∇ that generally admits torsion interacts with both g and L on an almost (para-)Hermitian manifold $(\mathfrak{M}, g, L)$, where L denotes either an almost complex structure J with $J^2 = -\mathrm{id}$ or an almost para-complex structure K with $K^2 = \mathrm{id}$. We show that ∇ becomes (para-)holomorphic and L becomes integrable if and only if the pair (∇, L) satisfies a torsion coupling condition. We investigate (para-)Hermitian manifolds $\mathfrak{M}$ in which this torsion coupling condition is satisfied by the following four connections (all possibly carrying torsion): $\nabla, \nabla^L, \nabla^*$, and $\nabla^\dagger = \nabla^{*L} = \nabla^{L*}$, where ∇^L and ∇^* are, respectively, L-conjugate and g-conjugate transformations of ∇. This leads to the following special cases (where T stands for torsion): (i) the case of $T = T^*, T^L = T^\dagger$, for which all four connections are Codazzi-coupled to g, but $d\omega \neq 0$, whence $\mathfrak{M}$ is called Codazzi-(para-)Hermitian; (ii) the case of $T = -T^\dagger, T^L = -T^*$, for which $d\omega = 0$, i.e., the manifold $\mathfrak{M}$ becomes (para-)Kähler. In the latter case, quadruples of (para-)holomorphic connections all with non-vanishing torsions may exist in (para-)Kähler manifolds, complementing the result of Fei and Zhang (Results Math 72:2037–2056, 2017) showing the existence of pairs of torsion-free connections, each Codazzi-coupled to both g and L, in Codazzi-(para-)Kähler manifolds.

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1. Introduction

On a differentiable manifold $\mathfrak{M}$ of even dimention, one can introduce three separate structures: an affine connection ∇, a pseudo-Riemannian metric g, and a tangent bundle isomorphism L. Compatibility of L with g defines an almost (para-)Hermitian manifold. Codazzi coupling of ∇ with g is well understood in affine geometry—a manifold $\mathfrak{M}$ equipped with a metric g and a torsion-free connection ∇ is called a *statistical manifold* if (g, ∇) is Codazzi-coupled [8]. Coupling of a torsion-free ∇ with L has also been studied (e.g. [6,11]). When a torsion-free connection ∇ is Codazzi-coupled to both g and L, then $(\mathfrak{M}, g, L)$ becomes a (para-)Kähler manifold [1].

In this paper, we relax the torsion-free assumption on affine connections, and investigate integrable structures that nevertheless are compatible with connections with torsion. Connections with torsion on almost (para-)Hermitian manifolds have always been a topic of interest, e.g. [3,14]. Integrability of L means that we are dealing with (para-)Hermitian manifolds.

Starting with ∇^L, the L-conjugate of ∇, we first define *torsion coupling* of (∇, L) as the following relation between the torsions T of ∇ and of ∇^L (Definition 2):

$$T^{\nabla}(LX, Y) = L(T^{\nabla^L}(X, Y)).$$

It is then shown (Theorem 6) that the (∇, L) pair is torsion-coupled if and only if both L is integrable and ∇ (and as a consequence, ∇^L as well) is a (para-)holomorphic connection. We then investigate, in the context of an almost (para-)Hermitian manifold, (para-)holomorphicity of ∇, ∇^L and ∇^*, Codazzi coupling of ∇ and of ∇^L with g, and their interactions. We show that given a (para-)Hermitian manifold $\mathfrak{M}$ with four connections ∇, ∇^L , ∇^*, $\nabla^\dagger = \nabla^{*L} = \nabla^{L*}$ (all possibly carrying torsion), if any one of them is torsion-coupled to L and any one is Codazzi-coupled to g, then in fact, all of them must be torsion-coupled to L and Codazzi-coupled to g (Theorem 13). This conclusion also holds when "torsion coupling" is replaced by the weaker condition of "(para-)holomorphicity" (Theorem 12). This leads to the definition of an almost Codazzi-(para-)Hermitian structure—which is an almost (para-)Hermitian manifold with all four connections being (para-)holomorphic and Codazzi-coupled to g. In this case, $T = T^*, T^L = T^\dagger$, but $d\omega \neq 0$ in general, unless the above torsions are all zero.

We next investigate, instead of Codazzi coupling of ∇ with g, special kinds of coupling of ∇ with the fundamental form (a non-degenerate 2-form) ω, while ∇ and L are torsion-coupled. In this case we derive the conditions on ∇ that will lead to $d\omega = 0$, hence turning $\mathfrak{M}$ into a (para-)Kähler manifold. Such (para-)Kähler manifolds may admit quadruples of (para-)holomorphic affine connections all with non-vanishing torsions: $T = -T^\dagger, T^L = -T^*$. Yet the quadruple of connections are neither Codazzi-coupled to L nor to g in general.

Because torsion coupling of ∇ with L is seen as a generalization of Codazzi coupling of torsion-free ∇ with L, our results above extend the findings of [1], showing (para-)Kähler manifold may admit a pair of torsion-free connections, $\nabla = \nabla^{*L}$ and $\nabla^L = \nabla^*$, in the so-called Codazzi-(para-)Kähler structure. A preliminary report of a subset of our results (without proof) appeared in [4].

2. (Para-)Holomorphic Pair (∇, L)

2.1. Splitting of $T\mathfrak{M}$ by L

For a smooth manifold $\mathfrak{M}$, an isomorphism L of the tangent bundle $T\mathfrak{M}$ is a smooth section of the bundle $\mathrm{End}(T\mathfrak{M})$ such that it is invertible everywhere. By definition, L is called an *almost complex structure* if $L^2 = -\mathrm{id}$, or an *almost para-complex structure* if $L^2 = \mathrm{id}$ and the multiplicities of the eigenvalues ± 1 are equal. We will use J and K to denote almost complex structures and almost para-complex structures, respectively, and use L when these two structures can be treated in a unified way. It is clear from our definitions that such structures exist only when $\mathfrak{M}$ is of even dimension.

Denote eigenvalues of L as $\pm\alpha$, where $\alpha = 1$ for $L = K$ and $\alpha = i$ for $L = J$, respectively. Following the standard procedure, we (para-)complexify $T\mathfrak{M}$ by tensoring with $\mathbb{C}$ or with the para-complex (also known as split-complex) field $\mathbb{D}$, and use $T^L\mathfrak{M}$ to denote the resulting $T\mathfrak{M} \otimes \mathbb{C}$ or $T\mathfrak{M} \otimes \mathbb{D}$, depending on the type of L. In analogy with standard notation in the complex case, let $T^{(1,0)}\mathfrak{M}$ and $T^{(0,1)}\mathfrak{M}$ be the eigenbundles of L corresponding to the eigenvalues $\pm\alpha$, i.e., at each point $p \in \mathfrak{M}$, the fiber is defined by

$$T^{(1,0)}(p) := \{X \in T_p^L\mathfrak{M} : L_p(X) = \alpha X\},$$
$$T^{(0,1)}(p) := \{X \in T_p^L\mathfrak{M} : L_p(X) = -\alpha X\}.$$

We will refer to vectors to be of type $(1,0)$ or $(0,1)$ if they take values in $T^{(1,0)}\mathfrak{M}$ or $T^{(0,1)}\mathfrak{M}$, respectively. Moreover, define $\pi^{(1,0)}$ and $\pi^{(0,1)}$ to be the projections of a vector field to $T^{(1,0)}\mathfrak{M}$ and $T^{(0,1)}\mathfrak{M}$ respectively.

The Nijenhuis tensor N_L associated with L is defined as

$$N_L(X,Y) := -L^2[X,Y] + L[X,LY] + L[LX,Y] - [LX,LY], \tag{1}$$

which satisfies

$$N(LX,LY) = L^2 N(X,Y), \qquad N(LX,Y) = N(X,LY).$$

When $N_L = 0$, the operator L is said to be integrable. As sub-bundles of the (para-)complexified tangent bundle $T^L\mathfrak{M}$, $T^{(1,0)}\mathfrak{M}$ and $T^{(0,1)}\mathfrak{M}$ are distributions. A distribution is called a foliation if it is closed under the bracket $[\cdot,\cdot]$. It is well-known that both $T^{(1,0)}\mathfrak{M}$ and $T^{(0,1)}\mathfrak{M}$ are foliations if and only if L is integrable, i.e., the integrability condition $N_L = 0$ is satisfied.

2.2. L-Conjugate of ∇

Starting from a (not necessarily torsion-free) connection ∇ operating on sections of $T\mathfrak{M}$, we can apply an L-conjugate transformation to obtain a new connection $\nabla^L := L^{-1}(\nabla(L(\ )))$, or

$$\nabla^L_X Y = L^{-1}(\nabla_X(LY)) \tag{2}$$

for any vector fields X and Y; here L^{-1} denotes the inverse isomorphism of L. It can be verified that indeed ∇^L is an affine connection. Furthermore,

$$(\nabla^{L_1})^{L_2} = \nabla^{L_1 \circ L_2},$$

so L-conjugation forms a group, with operator composition $\circ$ as group multiplication, and $L = \mathrm{id}$ the identity transformation as the group identity element.

Define a vector-valued skew-symmetric bilinear form S, a $(1, 2)$-tensor, via the expression

$$S(X, Y) := (\nabla_X L)Y - (\nabla_Y L)X, \tag{3}$$

where

$$(\nabla_X L)Y = \nabla_X(LY) - L(\nabla_X Y).$$

The pair (∇, L) is said to be *Codazzi-coupled* if $S = 0$.

Recall that the torsion T^∇ of ∇ is defined as:

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{4}$$

We easily derive

$$S(X, Y) = L(T^{\nabla^L}(X, Y) - T^\nabla(X, Y)). \tag{5}$$

This leads to the following well-known results.

Proposition 1. *(e.g., [12]) Let ∇ be an affine connection, and let L be an arbitrary tangent bundle isomorphism. Then the following statements are equivalent:*

(a) (∇, L) *is Codazzi-coupled.*
(b) $T^\nabla(X, Y) = T^{\nabla^L}(X, Y)$.
(c) (∇^L, L^{-1}) *is Codazzi-coupled.*

Corollary 1. *For the special case of (para-)complex operators $L^2 = \pm id$,*

(i) $\nabla^L = \nabla^{L^{-1}}$, *i.e., L-conjugate transformation is involutive, $(\nabla^L)^L = \nabla$.*
(ii) (∇, L) *is Codazzi-coupled if and only if (∇^L, L) is Codazzi-coupled.*

To appreciate the change of torsion via L-conjugate transformation, let us recall that, as an affine connection, ∇ gives rise to a map

$$\nabla : \Omega^0(T\mathfrak{M}) \to \Omega^1(T\mathfrak{M}),$$

where $\Omega^i(T\mathfrak{M})$ is the space of smooth i-forms with values in $T\mathfrak{M}$. We may extend this to a map

$$d^\nabla : \Omega^i(T\mathfrak{M}) \to \Omega^{i+1}(T\mathfrak{M})$$

by

$$d^{\nabla}(\alpha \otimes v) = d\alpha \otimes v + (-1)^i \alpha \wedge \nabla v$$

for any i-form α and vector field v. In the case that ∇ is flat, then $(d^{\nabla})^2 = 0$ and we get a chain complex whose cohomology is the de Rham cohomology twisted by the local system determined by ∇. Regarding L as an element of $\Omega^1(T\mathfrak{M})$, it is easy to check using local coordinates that

$$\begin{aligned}(d^{\nabla} L)(X,Y) &= (\nabla_X L)Y - (\nabla_Y L)X + LT^{\nabla}(X,Y)\\ &= S(X,Y) + LT^{\nabla}(X,Y)\\ &= LT^{\nabla^L}(X,Y).\end{aligned}$$

For later convenience, we introduce

Definition 2. A pair (∇, L) is said to be *torsion-coupled* if the following holds:

$$S(X,Y) = T^{\nabla}(LX,Y) - LT^{\nabla}(X,Y). \tag{6}$$

Equivalently, the torsion coupling condition can be written as

$$T^{\nabla}(LX,Y) = L(T^{\nabla^L}(X,Y)) \tag{7}$$

where ∇^L is the L-conjugate transform of ∇.

Proposition 2. *Suppose (∇, L) is torsion-coupled. Then*

(i) *(∇^L, L) is torsion-coupled.*
(ii) *T^{∇} satisfies $T^{\nabla}(LX,Y) = T^{\nabla}(X,LY)$.*
(iii) *T^{∇^L} satisfies $T^{\nabla^L}(LX,Y) = T^{\nabla^L}(X,LY)$.*

Proof. To prove (i), we note that Equation (7) can be re-written as

$$T^{\nabla^L}(LX,Y) = L(T^{\nabla}(X,Y)). \tag{8}$$

To prove (ii), we use Equation (7)

$$T^{\nabla}(LX,Y) = -L(T^{\nabla^L}(Y,X)) = -T^{\nabla}(LY,X) = T^{\nabla}(X,LY).$$

Statement (iii) then follows from (i) and (ii). □

We remark that the Codazzi coupling of and torsion coupling of (∇, L) are two different concepts that in general do not imply each other. However we have

Proposition 3. *Suppose $T^{\nabla} = 0$, then (∇, L) is Codazzi-coupled if and only if (∇, L) is torsion-coupled.*

Therefore torsion coupling can be viewed as a relaxation of the Codazzi coupling of ∇ with L when ∇ carries torsion. The pair (∇, L) can be both Codazzi-coupled and torsion-coupled when $T^{\nabla} = 0$.

2.3. Integrability of L

In Lemma 2.5 of [1], an expression for $N_L\left(X,Y\right)$ in terms of T^{∇} has been derived assuming $S=0$, i.e., Codazzi coupling. Here, we extend this to the case $S\neq 0$ and derive an expression of $N_L\left(X,Y\right)$ for an arbitrary S. We then show that when (∇, L) is torsion-coupled rather than Codazzi-coupled, then L is still integrable.

Lemma 3. *Given a connection ∇ with torsion T^{∇}, the Nijenhuis tensor N_L of a (para-)complex operator L is given by*

$$\begin{aligned} N_L\left(X,Y\right) &= (T^{\nabla^L}(L^2X,Y) - LT^{\nabla}(LX,Y)) \\ &\quad + (T^{\nabla^L}\left(LX,LY\right) - LT^{\nabla}(X,LY)). \end{aligned} \tag{9}$$

Proof. From (3) we have

$$\begin{aligned} \nabla_X\left(LY\right) - \nabla_Y\left(LX\right) &= \left(\nabla_X L\right)Y + L\nabla_X Y - \left(\nabla_Y L\right)X - L\nabla_Y X \\ &= S\left(X,Y\right) + L\left[X,Y\right] + LT\left(X,Y\right). \end{aligned}$$

Thus,

$$\begin{aligned} \nabla_X\left(LY\right) - \nabla_Y\left(LX\right) - L\left[X,Y\right] &= LT\left(X,Y\right) + S\left(X,Y\right) \\ &= LT^{\nabla^L}(X,Y). \end{aligned} \tag{10}$$

Now, replacing X by LY and Y by LX, and using the fact that L^2 is constant, we find

$$\begin{aligned} L^2\left(\nabla_{LY}X - \nabla_{LX}Y\right) - L\left[LY,LX\right] &= LT\left(LY,LX\right) + S\left(LY,LX\right) \\ &= LT^{\nabla^L}(LX,LY). \end{aligned} \tag{11}$$

Now multiply (10) by L^2 and subtract it from (11), the left-hand side becomes

$$\begin{aligned} &L^2\left(\nabla_{LY}X - \nabla_X\left(LY\right) + \nabla_Y\left(LX\right) - \nabla_{LX}Y\right) + L^3\left[X,Y\right] - L\left[LY,LX\right] \\ &\quad = L^2\left(\left[Y,LX\right] + T^{\nabla}(Y,LX) - \left[X,LY\right] - T^{\nabla}(X,LY)\right) + L^3\left[X,Y\right] \\ &\qquad - L\left[LY,LX\right] \end{aligned}$$

and the right-hand side is

$$LT^{\nabla^L}\left(LX,LY\right) - L^3T^{\nabla^L}\left(X,Y\right).$$

Multiplying both sides by $-L^{-1}$ and rearranging, we find

$$\begin{aligned} &-L^2[X,Y] + L[X,LY] + L[LX,Y] - [LX,LY] \\ &\quad = -T^{\nabla^L}\left(LY,LX\right) + L^2T^{\nabla^L}\left(X,Y\right) + LT^{\nabla}(Y,LX) - LT^{\nabla}(X,LY). \end{aligned}$$

Comparing with the definition of $N_L\left(X,Y\right)$ given by (1) yields (9). $\square$

From this Lemma, we immediately have

Proposition 4. *An almost (para-)complex operator L is integrable if there exists an affine connection ∇ such that (∇, L) is torsion-coupled.*

2.4. (Para-)Holomorphicity of ∇

The (para-)Dolbeault operator $\bar{\partial}$ for a given L on $T^L\mathfrak{M}$ is defined as [3]

$$\bar{\partial}_X Y = \frac{1}{4}\left([X,Y] - L^2[LX,LY] - L^{-1}[LX,Y] + L^{-1}[X,LY]\right) \tag{12}$$

for any vector fields X and Y. It can be checked easily that this expression is tensorial in X, that is $\bar{\partial}_{fX}Y = f\left(\bar{\partial}_X Y\right)$ and is a derivation. In the case when $L = J$, this defines the *pseudo-holomorphic structure* on $T^{\mathbb{C}}\mathfrak{M}$ and locally defines the differentiation of vector fields of type $(1,0)$ with respect to the anti-holomorphic coordinates $\frac{\partial}{\partial \bar{z}^i}$. It is also known as the *intrinsic Cauchy-Riemann operator* of J, since it only depends on the almost-complex structure. If J is integrable, then $\bar{\partial}$ defines a *holomorphic structure* on $T^{\mathbb{C}}\mathfrak{M}$. Similarly, if $L = K$ is integrable, then $\bar{\partial}$ defines para-holomorphic structure on $T^{\mathbb{D}}\mathfrak{M}$. For convenience and brevity, we will say that $\bar{\partial}$ defines a *(para-)holomorphic* structure independent of whether L is integrable or not (to avoid using the more appropriate yet cumbersome term of *pseudo-(para-)holomorphic* structure). However we will eventually specialize to the case of integrable L, so there will be no ambiguity. Using definition (1) for N_L, we can rewrite the definition (12) of $\bar{\partial}$ as

$$\bar{\partial}_X Y = \frac{1}{2}\left([X,Y] - L^{-1}[LX,Y] + \frac{1}{2}L^{-2}N_L(X,Y)\right). \tag{13}$$

From (12) we obtain that if X and Y are of the same type, then $\bar{\partial}_X Y = 0$. However, if X and Y are of the opposite types, then

$$\begin{aligned}\bar{\partial}_X Y &= \pi^{(1,0)}[X,Y] \quad \text{when}\, Y \in T^{(1,0)}\mathfrak{M} \text{ and } X \in T^{(0,1)}\mathfrak{M},\\ \bar{\partial}_X Y &= \pi^{(0,1)}[X,Y] \quad \text{when}\, Y \in T^{(0,1)}\mathfrak{M} \text{ and } X \in T^{(1,0)}\mathfrak{M}.\end{aligned}$$

Equivalently, note that if $Y \in T^{(1,0)}\mathfrak{M}$, then $\bar{\partial}Y$ is a vector-valued 1-form, of type $(1,0)$ as a vector and type $(0,1)$ as a 1-form, and conversely if $Y \in T^{(0,1)}\mathfrak{M}$. It is important to note that in conventional usage, when J is an integrable complex structure, $\bar{\partial}$ is usually restricted to $T^{(1,0)}\mathfrak{M}$, and induces a holomorphic vector bundle structure on $T^{(1,0)}\mathfrak{M}$. The corresponding operator on the anti-holomorphic sub-bundle $T^{(0,1)}\mathfrak{M}$ is usually denoted by ∂. However, if L is not necessarily integrable, it makes sense to consider $\bar{\partial}$ on the whole (para-)complexified tangent bundle $T^L\mathfrak{M}$—in that case $\bar{\partial}$ restricted to $T^{(0,1)}\mathfrak{M}$ will actually coincide with the operator that would usually be denoted by ∂.

Given a connection ∇ operating on $T^L\mathfrak{M}$, we can ask the question whether ∇ is compatible with $\bar{\partial}$. To understand this we may define an alternative operator $\bar{\partial}^{\nabla}$, which for $Y \in T^{(1,0)}\mathfrak{M}$ is defined as taking the $(0,1)$-part of the vector-valued 1-form ∇Y (and conversely on $T^{(0,1)}\mathfrak{M}$). This can be expressed as

$$\bar{\partial}^{\nabla}_X Y = \frac{1}{2}\left(\nabla_X Y - \nabla_{LX}\left(L^{-1}Y\right)\right) \tag{14}$$

for any vector fields X and Y in $T^L\mathfrak{M}$. This definition is a generalization of a similar operator that was defined in [3] for ∇ that is compatible with J. Clearly, $\bar{\partial}^{\nabla}_X Y = 0$ if X and Y are of the same type, and $\bar{\partial}^{\nabla}_X Y = \nabla_X Y$ is just the covariant derivative if X and Y are of the opposite types.

On a (para-)holomorphic vector bundle, a connection is conventionally called *(para-)holomorphic* if these two Dolbeault operators coincide. We extend this notion to arbitrary connections on $T^L\mathfrak{M} \cong T^{(1,0)}\mathfrak{M} \oplus T^{(0,1)}\mathfrak{M}$ that do not necessarily preserve $T^{(1,0)}\mathfrak{M}$ and $T^{(0,1)}\mathfrak{M}$.

Definition 4. (i). A connection ∇ is called (para-)holomorphic if $\bar{\partial}^{\nabla}_X Y = \bar{\partial}_X Y$ for any vector fields X and Y.

(ii). Two connections ∇^1 and ∇^2 are said to be $\bar{\partial}$-balanced when $\bar{\partial}^{\nabla^1}_X Y = \bar{\partial}^{\nabla^2}_X Y$ holds.

Note that the same caveat applies as for the operator $\bar{\partial}$ defined above: $\bar{\partial}^{\nabla}$ is defined on the full (para-)complexified tangent bundle $T^L\mathfrak{M}$, and therefore a restriction of it to $T^{(0,1)}\mathfrak{M}$ would actually be denoted by ∂^{∇} if defined on $T^{(0,1)}\mathfrak{M}$ as a (para-)holomorphic bundle.

It turns out that the (para-)holomorphicity property of an arbitrary connection is invariant with respect to L-conjugation.

Proposition 5. *∇^L is (para-)holomorphic if and only if ∇ is (para-)holomorphic.*

Proof. Since $\left(\nabla^L\right)^L = \nabla$, it is enough to show that if ∇ is (para-)holomorphic, so is ∇^L. Then,

$$\begin{aligned}
\bar{\partial}^{\nabla^L}_X Y &= \frac{1}{2}\left(\nabla^L_X Y - \nabla^L_{LX}\left(L^{-1}Y\right)\right) = \frac{1}{2}L^{-1}\left(\nabla_X\left(LY\right) - \nabla_{LX}Y\right)\\
&= -\frac{1}{2}L^{-1}\left(\nabla_{LX}Y - \nabla_{L(LX)}\left(L^{-1}Y\right)\right)\\
&= -L^{-1}\bar{\partial}^{\nabla}_{LX}Y. \qquad (15)
\end{aligned}$$

If $\bar{\partial}^{\nabla} = \bar{\partial}$, then using the expression (13) for $\bar{\partial}$, we get

$$\begin{aligned}
\bar{\partial}^{\nabla^L}_X Y &= -L^{-1}\bar{\partial}_{LX}Y\\
&= -\frac{1}{2}L^{-1}\left(\left[LX,Y\right] - L^{-1}\left[L^2X,Y\right] + \frac{1}{2}L^{-2}N_L\left(LX,Y\right)\right)\\
&= \frac{1}{2}\left(\left[X,Y\right] - L^{-1}\left[LX,Y\right] + \frac{1}{2}L^{-2}N_L\left(X,Y\right)\right) \qquad (16)\\
&= \bar{\partial}_X Y \qquad (17)
\end{aligned}$$

since L^2 is constant and $L^{-1}N_L\left(LX,Y\right) = -N_L\left(X,Y\right)$. □

Proposition 5 implies that a sufficient condition for (∇, ∇^L) to be $\bar{\partial}$-balanced is that either of them is (para-)holomorphic. (Para-)holomorphicity of ∇ (and ∇^L) is stronger than (∇, ∇^L) being $\bar{\partial}$-balanced.

Using (12) and (14), we can also derive a necessary and sufficient condition for (para-)holomorphicity in terms of N_L, T^∇, and T^{∇^L}.

Lemma 5. *Given an arbitrary pair (∇, L) on a manifold, the connection ∇ is (para-)holomorphic if and only if*

$$\frac{1}{2}L^2 N_L(LX,Y) = T^\nabla(LX,Y) - LT^{\nabla^L}(X,Y). \tag{18}$$

Proof. Using the expression (13) for $\bar\partial$ we get

$$\begin{aligned}\bar\partial_X Y &= \frac{1}{2}\left([X,Y] - L^{-1}[LX,Y] + \frac{1}{2}L^{-2}N_L(X,Y)\right)\\ &= \frac{1}{2}\left(\nabla_X Y - \nabla_Y X - T^\nabla(X,Y)\right)\\ &\quad - \frac{1}{2}L^{-1}\left(\nabla_{LX}Y - \nabla_Y(LX) - T^\nabla(LX,Y)\right) + \frac{1}{4}L^{-2}N_L(X,Y) \end{aligned} \tag{19}$$

where we have also used the definition (4) of the torsion T. Then equating (19) and (14), and then simplifying, gives

$$\begin{aligned}&-\left(\nabla_{LX}L^{-1}\right)Y - \left(L^{-1}\nabla_Y L\right)X\\ &\quad = -T^\nabla(X,Y) + L^{-1}T^\nabla(LX,Y) + \frac{1}{2}L^{-2}N_L(X,Y).\end{aligned} \tag{20}$$

However, $\left(L^{-1}\nabla_Y L\right) = -\left(\nabla_Y L^{-1}\right)L$ and moreover, $L^{-1} = L^{-2}L$, where $L^{-2} = \pm 1$ is constant, so the left-hand side of (20) becomes $-L^{-2}S(LX,Y)$. Therefore, we can simplify (20) to

$$S(LX,Y) = L^2T^\nabla(X,Y) - LT^\nabla(LX,Y) - \frac{1}{2}N_L(X,Y). \tag{21}$$

Replacing X by $L^{-1}X = L^{-2}LX$ in (21) gives us

$$S(X,Y) = T^\nabla(LX,Y) - LT^\nabla(X,Y) - \frac{1}{2}L^2N_L(LX,Y).$$

Rearranging and using (5) gives (18). □

From Lemma 3 and Lemma 5 , we prove one of the main theorems of our paper.

Theorem 6. *A pair (∇, L) on a manifold is torsion-coupled (as in Definition 2) if and only if the connection ∇ is (para-)holomorphic and L is integrable.*

Proof. Clearly, if ∇ is (para-)holomorphic and L is integrable, then by Lemma 5, equation (18) is satisfied. Since L is integrable, $N_L = 0$. Hence the right-hand side of (18) is zero, which gives (8), the condition that (∇, L) being torsion-coupled. Conversely, assume (8) is satisfied. From Lemma 3, $N_L = 0$ and hence L is integrable. Moreover, equation (18) is satisfied trivially. So by Lemma 5, the connection ∇ is (para-)holomorphic. □

The significance of Theorem 6 is that it gives a generalization of the Codazzi coupling condition for L that was used in [1] in the case $T^{\nabla} = 0$. In fact, it follows immediately that if $T^{\nabla} = 0$ then Codazzi coupling of ∇ with L makes L integrable *and* makes ∇ (para-)holomorphic. Recall from Proposition 1 that (∇, L) is a Codazzi pair if and only if $T^{\nabla} = T^{\nabla^L}$. Theorem 6 gives a modification of this for the situation where (∇, L) is torsion-coupled, but not necessarily a Codazzi-coupled.

Combining Proposition 5 with Theorem 6, we obtain

Proposition 6. *A pair (∇, L) is torsion-coupled if and only if the pair (∇^L, L) is torsion-coupled.*

2.5. Properties of (Para-)Holomorphic Connections

Introduce

$$\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^L),$$

which satisfies

$$\tilde{\nabla} L \equiv 0.$$

A connection with respect to which L is parallel is called a *(para-)complex connection*, and in particular, such a connection preserves the decomposition $T^L\mathfrak{M} \cong T^{(1,0)}\mathfrak{M} \oplus T^{(0,1)}\mathfrak{M}$. So starting from any connection ∇, we can construct its conjugate ∇^L, and the average of the two is the (para-)complex connection $\tilde{\nabla}$.

Now, define $\theta\,(X, Y)$ to be a vector-valued bilinear form given by

$$\theta(X, Y) = \frac{1}{2}(\nabla^L_X Y - \nabla_X Y) = \frac{1}{2} L^{-1}(\nabla_X L)Y. \tag{22}$$

We can then write

$$\nabla = \tilde{\nabla} - \theta, \qquad \nabla^L = \tilde{\nabla} + \theta.$$

The quantity θ therefore measures the failure of both ∇ and ∇^L to be a (para-)complex connection. This situation is analogous to the relationship between Levi-Civita connection and the pair of g-conjugate connections ∇, ∇^*; see Sect. 3.2 below. Conjugacy of ∇ and ∇^L mirrors conjugacy of ∇, ∇^*.

Proposition 7. *The vector-valued bilinear form $\theta(X, Y)$*

(i) *always satisfies $L\theta\,(X, Y) + \theta\,(X, LY) = 0$;*
(ii) *satisfies $\theta(X, Y) = \theta(Y, X)$ if and only if (∇, L), or equivalently (∇^L, L), is Codazzi-coupled;*
(iii) *satisfies $\theta(LX, Y) = \theta(X, LY)$ if any only if $\bar{\partial}^{\nabla} = \bar{\partial}^{\nabla^L}$, i.e., ∇ and ∇^L are $\bar{\partial}$-balanced.*

Proof. To show (i),

$$\begin{aligned}\theta(X, LY) &= \frac{1}{2}(\nabla^L_X(LY) - \nabla_X(LY)) \\ &= \frac{1}{2}\left(L^{-1}\nabla_X(L^2Y) - L\nabla^L_X Y\right) \\ &= \frac{1}{2}L\left(\nabla_X Y - \nabla^L_X Y\right) \\ &= -L\theta(X, Y).\end{aligned}$$

To show (ii), one only needs to note

$$\theta\left(X, Y\right) - \theta\left(Y, X\right) = \frac{1}{2}L^{-1}\left(S\left(X, Y\right)\right). \tag{23}$$

To show (iii), we use $\nabla^L = \nabla + 2\theta$ to write

$$\begin{aligned}\bar{\partial}^{\nabla^L}_X Y &= \frac{1}{2}\left(\nabla^L_X Y - \nabla^L_{LX}(L^{-1}Y)\right) \\ &= \frac{1}{2}\left((\nabla_X Y + 2\theta(X, Y)) - (\nabla_{LX}(L^{-1}Y) + 2\theta(LX, L^{-1}Y))\right) \\ &= \bar{\partial}^{\nabla}_X Y + \theta(X, Y) - \theta(LX, L^{-1}Y).\end{aligned}$$

Hence, $\bar{\partial}^{\nabla^L}_X Y = \bar{\partial}^{\nabla}_X Y$ if only if $\theta(X, Y) = \theta(LX, L^{-1}Y)$. □

Proposition 5 says that $\bar{\partial}^{\nabla} = \bar{\partial}$ if and only if $\bar{\partial}^{\nabla^L} = \bar{\partial}$. Proposition 7 part (iii) further says that whenever $\bar{\partial}^{\nabla} = \bar{\partial}^{\nabla^L}$, then

$$L\theta\left(X, Y\right) = -\theta\left(X, LY\right) = -\theta\left(LX, Y\right) = L^{-1}\theta\left(LX, LY\right). \tag{24}$$

Taken together, the relations (24) hold whenever ∇ is (para-)holomorphic.

For a (para-)holomorphic ∇, $\theta\left(X, Y\right)$ vanishes whenever X and Y are of different types. Moreover, if X and Y are both of type $(1, 0)$, $\theta\left(X, Y\right)$ is of type $(0, 1)$, and vice versa.

From Theorem 6, whenever (∇, L) is torsion-coupled, ∇ is (para-)holomorphic (in addition to L being integrable). In other words, we expect (6) will lead to (24). To see it directly, we can readily verify from (6) that

$$S(LX, Y) = -LS(X, Y).$$

Therefore

$$\theta(LX, Y) - \theta(Y, LX) = -L(\theta(X, Y) - \theta(Y, X)),$$

which yields $\theta(LX, Y) = -L\theta(X, Y)$ upon using (i) of Proposition 7.

To conclude, torsion coupling of (∇, L) (and equivalently of (∇^L, L)) implies (para-)holomorphicity of ∇ (and equivalently of ∇^L), which implies ∇ and ∇^L are $\bar{\partial}$-balanced.

3. Interaction of ∇ with Almost (Para-)Hermitian Structures (g, L)

3.1. Almost (Para-)Hermitian Structures

The compatibility condition between g and an almost (para-)complex structure $J(K)$ is well-known. We say that g is compatible with J if J is orthogonal, i.e.,

$$g(JX, JY) = g(X, Y) \tag{25}$$

holds for any vector fields X and Y. Similarly we say that g is compatible with K if

$$g(KX, KY) = -g(X, Y) \tag{26}$$

is always satisfied, which implies that g must be of split signature. When expressed using L, (25) and (26) have the common form

$$g(X, LY) + g(LX, Y) = 0. \tag{27}$$

When specified in terms of compatible g and L, the manifold $(\mathfrak{M}, g, L)$ is said to be *almost (para-)Hermitian*, and (para-)Hermitian if L is integrable.

For any almost (para)-Hermitian manifold, we can define the 2-form $\omega(X, Y) = g(LX, Y)$, called the *fundamental form*, which turns out to satisfy $\omega(X, LY) + \omega(LX, Y) = 0$. The three structures, a pseudo-Riemannian metric g, a nondegenerate 2-form ω, and a tangent bundle isomorphism $L : T\mathfrak{M} \to T\mathfrak{M}$ form a "compatible triple" such that given any two, the third one is uniquely specified; the triple is rigidly "interlocked".

3.2. g-Conjugate Connection and Codazzi Coupling

Given a pair (∇, g), we define the $(0, 3)$-tensor C by

$$C(X, Y, Z) := (\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y). \tag{28}$$

When $C = 0$, we say g is parallel under ∇.

Given a pair (∇, g), we can also construct ∇^*, called the g-conjugate connection, by

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y). \tag{29}$$

It can be checked easily that (i) ∇^* is indeed a connection and (ii) g-conjugation of a connection is involutive, i.e., $(\nabla^*)^* = \nabla$.

These two constructions from an arbitrary pair (∇, g) are related via

$$C(X, Y, Z) = g((\nabla^* - \nabla)_Z X, Y), \tag{30}$$

so that

$$\begin{aligned} g(\nabla^*_Z X, Y) &= g(\nabla_Z X, Y) + C(X, Y, Z), \\ C^*(X, Y, Z) := (\nabla^*_Z g)(X, Y) &= -C(X, Y, Z). \end{aligned}$$

Therefore $C(X,Y,Z) = C^*(X,Y,Z) = 0$ if and only if $\nabla^* = \nabla$, that is, ∇ is g-self-conjugate. A connection that is both g-self-conjugate and torsion-free is, by definition, the Levi-Civita connection ∇^{LC} associated to g.

A simple calculation reveals that

$$C(X,Y,Z) - C(Z,Y,X) = g(T^{\nabla^*}(Z,X) - T^{\nabla}(Z,X),Y), \tag{31}$$

where T^{∇} denotes the torsion of ∇ given in (4), and T^{∇^*} the torsion for ∇^*.

We say that g and ∇ are *Codazzi-coupled* when

$$(\nabla_Z g)(X,Y) = (\nabla_X g)(Z,Y), \tag{32}$$

or equivalently,

$$C(X,Y,Z) = C(Z,Y,X).$$

Note that $C(X,Y,Z) = C(Y,X,Z)$ always holds, due to $g(X,Y) = g(Y,X)$. Therefore, imposing Codazzi coupling condition leads to the total symmetry of $C(X,Y,Z)$ in all its three entries. These well-known facts are summarized in the following Proposition.

Proposition 8. *Let g be a pseudo-Riemannian metric, ∇ an arbitrary affine connection, and ∇^* be the g-conjugate connection of ∇. Then the following statements are equivalent:*

(a) (∇, g) *is Codazzi-coupled;*
(b) (∇^*, g) *is Codazzi-coupled;*
(c) C *is totally symmetric;*
(d) C^* *is totally symmetric;*
(e) $T^{\nabla} = T^{\nabla^*}$.

3.3. Compatibility of ∇^L with g

Lemma 7. *On an almost (para-)Hermitian manifold $(\mathfrak{M}, g, L)$, the following two identities hold:*

$$C^L(LX,Y,Z) + C(X,LY,Z) = 0, \tag{33}$$

where $C^L(X,Y,Z) := (\nabla^L_Z g)(X,Y)$, and

$$C(X,Y,Z) + L^{-2}C(LX,LY,Z) = 2g(\theta(Z,X),Y) + 2g(\theta(Z,Y),X). \tag{34}$$

Proof. By definition,

$$\begin{aligned}(\nabla_X g)(Y,LZ) &= X(g(Y,LZ)) - g(\nabla_X Y, LZ) - g(Y, \nabla_X(LZ));\\ (\nabla^L_X g)(LY,Z) &= X(g(LY,Z)) - g(\nabla^L_X(LY),Z) - g(LY,\nabla^L_X Z)\\ &= X(g(LY,Z)) - g(L(\nabla_X Y),Z) - g(LY, L^{-1}\nabla_X(LZ)).\end{aligned}$$

Therefore,

$$\begin{aligned}(\nabla^L_X g)(LY,Z) + (\nabla_X g)(Y,LZ) &= X(g(Y,LZ) + g(LY,Z))\\ &\quad -(g(\nabla_X Y, LZ) + g(L(\nabla_X Y),Z)) - (g(Y,\nabla_X(LZ))\\ &\quad +g(LY, L^{-1}(\nabla_X(LZ))))\\ &= 0,\end{aligned}$$

where each of the three parentheses evaluates to zero on account of (27). So (33) is obtained as being exactly

$$(\nabla^L_X g)(LY, Z) + (\nabla_X g)(Y, LZ) = 0. \tag{35}$$

Next, we recall $\nabla^L_X Y = \nabla_X Y + 2\theta(X, Y)$, and have

$$\begin{aligned} C^L(X, Y, Z) &= (\nabla^L_Z g)(X, Y) = Z(g(X, Y)) - g(\nabla^L_Z X, Y) - g(X, \nabla^L_Z Y) \\ &= Z(g(X, Y)) - g(\nabla_Z X + 2\theta(Z, X), Y) - g(X, \nabla_Z Y + 2\theta(Z, Y)) \\ &= C(X, Y, Z) - 2g(\theta(Z, X), Y) - 2g(X, \theta(Z, Y)). \end{aligned}$$

Combining with (33), we obtain (34). □

Lemma 8. (∇^L, g) *is Codazzi-coupled if and only if*

$$C(LX, Y, Z) = C(LZ, Y, X), \quad \textit{or equivalently} \tag{36}$$

$$C(X, LY, Z) = C(X, LZ, Y). \tag{37}$$

Proof. Codazzi coupling of (∇^L, g) means that $C^L(X, Y, Z)$ is totally symmetric. By identity (33), this means $C(L^{-1}X, LY, Z)$ is totally symmetric in X, Y, Z, such that

$$C(L^{-1}X, LY, Z) = C(L^{-1}X, LZ, Y).$$

Substituting $L^{-1}X$ with X yields (36). Switching the first and second entry and then renaming leads to the second equality. □

Proposition 9. *On an almost (para-)Hermitian manifold* $(\mathfrak{M}, g, L)$*, the following statements are equivalent:*

(a) $C(LX, Y, Z) = C(X, LY, Z)$.
(b) $C^L(X, Y, Z) = -C(X, Y, Z)$.
(c) C *and* θ *satisfy*

$$C\left(X, Y, Z\right) = g\left(\theta\left(Z, X\right), Y\right) + g\left(\theta\left(Z, Y\right), X\right); \tag{38}$$

(d) $\tilde{\nabla} = \frac{1}{2}\left(\nabla + \nabla^L\right)$ *is a (para-)complex-metric connection defined as* $\tilde{\nabla} L = \tilde{\nabla} g = 0$.

Proof. $(a) \iff (c)$. From (34), (c) is equivalent to $C(X, Y, Z) = C(L^{-1}X, LY, Z)$, which is (a) after substituting X by LX.

$(c) \iff (d)$. Recall that $\tilde{\nabla} = \nabla + \theta$, therefore we can write

$$\begin{aligned} \left(\tilde{\nabla}_Z g\right)\left(X, Y\right) &= \left(\nabla_Z g\right)\left(X, Y\right) - g\left(\theta\left(Z, X\right), Y\right) - g\left(\theta\left(Z, Y\right), Z\right) \\ &= C\left(X, Y, Z\right) - g\left(\theta\left(Z, X\right), Y\right) - g\left(\theta\left(Z, Y\right), Z\right). \end{aligned}$$

This vanishes if and only if (38) holds.

$(b) \iff (d)$. Statement (d), that is, $\tilde{\nabla} g = 0$ is equivalent to $\nabla^L g = -\nabla g$, which is (b) after substituting the definitions of C^L, C. □

Recall that on an almost (para-)Hermitian manifold $(\mathfrak{M}, g, L)$, a *(para-)complex-metric connection* is defined by $\nabla g = \nabla L = 0$. As a result, $\nabla \omega = 0$, $\nabla h = 0$ as well, where ω is the fundamental form and h the (para-)Hermitian form. An important family of such (para-)complex-metric connection is the so-called Gauduchon connections [3], which are (para-)complex ($\nabla L = 0$) and (para-)Hermitian ($\nabla h = 0$), but carry torsions. Gauduchon connections are not unique and form a line, which include the unique Chern connection (after imposing the additional condition of holomorphicity) and the unique Bismut connection (after imposing the additional condition of total skew-symmetry of its torsion).

With respect to $\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^L)$, $\tilde{\nabla} L = 0$ is always guaranteed, so Proposition 9 applies to the case when $\tilde{\nabla} g = 0$, that is, when $\tilde{\nabla}$ is Gauduchon. Note that $\nabla^L \neq \nabla^*$ in general; even though $\frac{1}{2}(\nabla + \nabla^*) g \equiv 0$, $\frac{1}{2}(\nabla + \nabla^*) L \neq 0$. It can be shown that $g(X, (\nabla^L - \nabla^*)_Z Y) \equiv K(X, Y, Z)$ may still be a (non-zero) skew-symmetric contorsion tensor.

3.4. Holomorphicity of ∇^*

We have seen in Proposition 5 that ∇ is (para-)holomorphic if and only if ∇^L is (para-)holomorphic, and in Proposition 8 that ∇ is Codazzi-coupled to g if and only if ∇^* is Codazzi-coupled to g. We now investigate the relationship between (para-)holomorphicity of ∇^* and Codazzi coupling of ∇^L with g.

Lemma 9. *Suppose g is a metric, ∇ is a connection, and ∇^* is the g-conjugate connection. Also, suppose L is an arbitrary almost (para-)complex structure. Then, (∇, ∇^*) is $\bar{\partial}$-balanced, i.e., $\bar{\partial}^{\nabla^*} = \bar{\partial}^{\nabla}$, if and only if*

$$C(LX, Y, Z) = C(X, Y, LZ) \tag{39}$$

for any vector fields X, Y, Z.

Proof. We need to show that $\bar{\partial}_Z^{\nabla^*} X = \bar{\partial}_Z^{\nabla} X$ if and only if (39) holds. From (14), and using (30), we have

$$\begin{aligned} g\left(\bar{\partial}_Z^{\nabla^*} X, Y\right) &= \frac{1}{2} g\left(\nabla_Z^* X, Y\right) - \frac{1}{2} g\left(\nabla_{LZ}^*\left(L^{-1} X\right), Y\right) \\ &= \frac{1}{2} g\left(\nabla_Z X, Y\right) - \frac{1}{2} g\left(\nabla_{LZ}\left(L^{-1} X\right), Y\right) + \frac{1}{2} C(X, Y, Z) \\ &\quad - \frac{1}{2} C\left(L^{-1} X, Y, LZ\right) \\ &= g\left(\bar{\partial}_Z^{\nabla} X, Y\right) + \frac{1}{2}\left(C(X, Y, Z) - L^{-2} C(LX, Y, LZ)\right). \end{aligned}$$

Hence, $\bar{\partial}^{\nabla^*} = \bar{\partial}^{\nabla}$ if and only if

$$C(X, Y, Z) = L^{-2} C(LX, Y, LZ). \tag{40}$$

Replacing X by LX in (40), and using the fact that $L^2 = \pm \mathrm{id}$, we obtain (39). $\square$

Lemma 10. *Suppose either* ∇ *or* ∇^* *is (para-)holomorphic. Then*

$$C(LX,Y,Z)+C(X,LY,Z)=C(X,Y,LZ)+L^{-2}C(LX,LY,LZ) \quad (41)$$

for any vector fields X, Y, Z.

Proof. When ∇ is (para-)holomorphic, i.e., $\bar{\partial}^{\nabla}=\bar{\partial}$, then $\bar{\partial}^{\nabla^L}=\bar{\partial}$ also holds, so by (iii) of Proposition 7, equations (24) hold. Therefore

$$\begin{aligned} g(\theta(LZ,L^{-1}X),Y) &= g(\theta(Z,L(L^{-1}X)),Y)=g(\theta(Z,X),Y),\\ g(\theta(LZ,Y),L^{-1}X) &= g(-L(\theta(Z,Y)),L^{-1}X)=g(\theta(Z,Y),L(L^{-1}X))\\ &= g(\theta(Z,Y),X). \end{aligned}$$

So

$$g(\theta(LZ,L^{-1}X),Y)+g(\theta(LZ,Y),L^{-1}X)=g(\theta(Z,X),Y)+g(\theta(Z,Y),X).$$

Because of the identity (34), this means

$$C(X,Y,Z)+C(LX,LY,L^{-2}Z)=C(L^{-1}X,Y,LZ)+C(X,LY,L^{-1}X);$$

that is, the left-hand side is invariant upon substituting $(X,Z)\to(L^{-1}X,LZ)$. Finally, substituting LX for X in the above equation yields (41).

With respect to ∇^*, we have similar expressions

$$\nabla^*_X Y=\frac{1}{2}L^{-1}(\nabla^*_X L)Y, \qquad C^*(X,Y,Z)=(\nabla^*_Z g)(X,Y).$$

They satisfy a similar equality (34) where $\theta\to\theta^*$ and $C\to C^*$. When ∇^* is (para-)holomorphic, i.e., $\bar{\partial}^{\nabla^*}=\bar{\partial}$, then we can derive

$$C^*(X,Y,Z)+C^*(LX,LY,L^{-2}Z)=C^*(L^{-1}X,Y,LZ)+C^*(X,LY,L^{-1}X).$$

Since $C^*(X,Y,Z)=-C(X,Y,Z)$, from the above we obtain equation (41) as well. $\square$

Summarizing Lemma 9, Proposition 9, Proposition 8, and Lemma 8, we have the following equalities on C:

(a) (∇,∇^*) is $\bar{\partial}$-balanced: $C(LX,Y,Z)=C(X,Y,LZ)$, or equivalently, $C(X,LY,Z)=C(X,Y,LZ)$;
(b) $\frac{1}{2}(\nabla+\nabla^L)$ is a (para-)complex-metric connection: $C(LX,Y,Z)=C(X,LY,Z)$;
(c) (∇,g) is Codazzi-coupled: $C(X,Y,Z)=C(Z,Y,X)$ or C is totally symmetric;
(d) (∇^L,g) is Codazzi-coupled: $C(LX,Y,Z)=C(LZ,Y,X)$, or equivalently $C(X,LY,Z)=C(X,LZ,Y)$.

We now state the following Theorem.

Theorem 11. *Let* $(\mathfrak{M},g,L)$ *be an almost (para-)Hermitian manifold with an affine connection* ∇. *Then, with respect to the four statements above,*

(i) *(a) implies (b).*
(ii) *(b) and (c) imply (d).*

(iii) *(b) and (d) imply (c).*
(iv) *(c) and (d) imply (a).*

When either ∇ *or* ∇^* *is (para-)holomorphic, i.e, (41) holds, then,*

(v) *(b) implies (a).*
(vi) *(c) implies (d), (a) and (b).*
(vii) *(d) implies (c), (b) and (a).*

Proof. (i) From (a), renaming $X \leftrightarrow Y$, we have $C(LY, X, Z) = C(Y, X, LZ)$. Symmetry of the first two entries of C yields

$$C(X, LY, Z) = C(X, Y, LZ) = C(LX, Y, Z),$$

which is what C obeys in (b). So we have proven (a) implies (b).

(ii) Due to (c), C is totally symmetric in the three entries. Therefore, (b) becomes $C(LX, Z, Y) = C(LY, Z, X)$. Renaming $Y \leftrightarrow Z$ yields (d).

(iii) Start from (d),

$$\begin{aligned} C(LZ, Y, X) &= C(LX, Y, Z) = C(X, LY, Z) \\ &= C(LY, X, Z) = C(Y, X, LZ) = C(X, Y, LZ) \end{aligned}$$

where the second equality used (b) and the fourth equality used (d). Rename LZ as Z, we have $C(Z, Y, X) = C(X, Y, Z)$ which is (c).

(iv) Switch the first and last entry on the right-hand side of (d), which is allowed due to (c), yields (a).

The proof of following statements relies on equation (41), which encodes the assumption that ∇ or ∇^* is (para-)holomorhpic.

(v) By statement (b)

$$C(LX, Y, Z) = C(X, LY, Z), \quad C(X, Y, LZ) = L^{-2}C(LX, LY, LZ).$$

Therefore, in equation (41), the two left-hand side terms are equal, and so are the two right-hand side terms, giving rise to

$$2C(LX, Y, Z) = 2C(X, Y, LZ),$$

which yields $C(LX, Y, Z) = C(X, Y, LZ)$. This is (a).

(vi) Starting from equation (41), substitute $(X, Y, Z) \leftrightarrow (Y, Z, X)$ to obtain

$$C(LY, Z, X) + C(Y, LZ, X) = C(Y, Z, LX) + L^{-2}C(LY, LZ, LX). \quad (42)$$

Sum (41) and (42), and then use total symmetry of C due to (c) yields

$$2C(X, LY, Z) = 2L^{-2}C(LX, LY, LZ),$$

which further reduces to

$$C(LX, Y, Z) = C(LZ, Y, X) = C(X, Y, LZ),$$

which are (d) and (a) respectively.

(vii) Substitute $X \to LX$ in (41),

$$L^2 C(X,Y,Z) + C(LX,LY,Z) = C(LX,Y,LZ) + C(X,LY,LZ). \tag{43}$$

Now, making use of (d),

$$C(LX,Y,LZ) = C(L^2 Z,Y,X) = L^2 C(Z,Y,X),$$
$$C(X,LY,LZ) = C(X,L^2 Z,Y) = L^2 C(X,Z,Y),$$

equation (43) becomes

$$C(X,Y,Z) + L^{-2} C(LY,LX,Z) = C(Z,Y,X) + C(X,Z,Y).$$

Exchanging $Y \leftrightarrow Z$ yields

$$C(X,Z,Y) = L^{-2} C(LZ,LX,Y) = C(Y,Z,X) + C(X,Y,Z).$$

Summing the above two equations, and taking into account

$$C(LY,LX,Z) = C(LZ,LX,Y) = C(LZ,LY,X)$$

due to (d), as well as the symmetry of C in its first two entries, we obtain

$$2L^{-2} C(LZ,LX,Y) = 2C(Z,Y,X).$$

This gives rise to

$$C(LZ,L^{-1}Y,X) = C(Z,Y,X),$$

which is (b), and

$$C(Z,Y,X) = L^{-2} C(LZ,LY,X) = L^{-2} C(LX,LY,Z) = C(X,Y,Z),$$

which is (c). □

Recall from Proposition 8 that the g-conjugate connection ∇^* is Codazzi-coupled to g if and only if ∇ is Codazzi-coupled to g. Thus, given a Codazzi-coupled pair (∇, g), we can uniquely extend it to a *Codazzi triple* (∇, ∇^*, g), namely, conjugate connections ∇, ∇^* that are both Codazzi-coupled to g. Part (i) to (iv) deals with the case $\bar{\partial}^{\nabla^*} = \bar{\partial}^{\nabla}$. They say that requiring these two connections ∇, ∇^* to be $\bar{\partial}$-balanced (in addition to their Codazzi coupling with g) amounts to requiring ∇^L to be Codazzi-coupled with g as well. In fact any two of the three statements (b), (c) and (d) imply the third. Part (v) to (vii) deals with the case of $\bar{\partial}^{\nabla} = \bar{\partial}$ or the case of $\bar{\partial}^{\nabla^*} = \bar{\partial}$, that is, when ∇ or ∇^* is (para-)holomorphic. In such situation, statements (a) and (b) are equivalent, whereas statements (c) and (d) are equivalent and each implies (a) and (b).

3.5. A Quadruple of Conjugate Connections

So far, given an affine connection ∇, we have only considered its L -conjugate connection ∇^L and the g-conjugate connection ∇^*. However, we may also apply g-conjugation to ∇^L and L-conjugation to ∇^* to obtain another affine connection. In fact, [1] shows that

Proposition 10. ([1], Theorem 2.13) *On an almost (para-)Hermitian manifold $(\mathfrak{M}, g, L)$, the g-conjugate ∇^* and L-conjugate ∇^L of any connection ∇ satisfies*

$$(\nabla^*)^L = (\nabla^L)^*. \tag{44}$$

Denote $\nabla^\dagger = (\nabla^*)^L = (\nabla^L)^*$, the set of transformations $\mathbb{Z}_4 = \{id, *, L, \dagger\}$ gives a 4-element Klein group that acts on affine connections.

On a (para-)Hermitian manifold $(\mathfrak{M}, g, L)$ we may in general have a family of four affine connections ∇ , ∇^*, ∇^L, and $\nabla^\dagger$. Each one may satisfy the torsion coupling property with respect to L or the Codazzi coupling property with respect to g. In general this gives eight possible coupling conditions about these connections. However, from Proposition 6, we know that a connection ∇ is torsion-coupled to L if and only if its L -conjugate connection ∇^L also satisfies this property. Moreover, Proposition 8 tells us that a connection ∇ is Codazzi-coupled to g if and only if its g-conjugate connection ∇^* also satisfies this property. Therefore, the eight coupling conditions about connections can be effectively reduce to four. Theorem 11 further relates the interaction of ∇^L with g to interaction of ∇^* with L. We can now state one of our main theorems.

Theorem 12. *Let $(\mathfrak{M}, g, L)$ be an almost (para-)Hermitian manifold with affine connections ∇, ∇^*, ∇^L, and $\nabla^\dagger$. If any one of them is (para-)holomorphic and any one of them is Codazzi-coupled to g, then all four connections are (para-) holomorphic as well as Codazzi-coupled to g. Furthermore,*

$$\left(\nabla + \nabla^L\right)/2 = \left(\nabla^* + \nabla^\dagger\right)/2 \ (=: \tilde{\nabla})$$

as its unique (para-)Chern connection

$$\tilde{\nabla} L = \tilde{\nabla}\omega = \tilde{\nabla} g = 0.$$

Proof. From Proposition 10, we know that $\mathbb{K}_4 = \{\mathrm{id}, *, L, \dagger\}$ gives a 4-element Klein group that acts on affine connections. By Proposition 8, ∇ is Codazzi-coupled to g if and only if ∇^* is, and ∇^L is Codazzi-coupled to g if and only if $\nabla^\dagger$ is. So the condition "any of the four connections is Codazzi-coupled to g" translates to the condition that "either (c) or (d) holds" with respect to the statements (a)–(d) referenced in Theorem 11.

By Proposition 5, ∇ is (para-)holomorphic if and only if ∇^L is, and ∇^* is (para-)holomorphic if and only if $\nabla^\dagger$ is. From (vi) and (vii) of Theorem 11, the validity of either (c) or (d) will lead to the other three statements of (a)–(d) to hold. Therefore, we conclude that so long as one of the four connections is (para-)holomophic and one of the four Codazzi-coupled to g, then all four connections are.

Finally, Proposition (9) tells us that $\frac{1}{2}(\nabla + \nabla^L)g = 0$. So $\frac{1}{2}(\nabla + \nabla^L)$ must be g-self-conjugate, that is

$$\frac{1}{2}\left(\nabla + \nabla^L\right) = \left(\frac{1}{2}\left(\nabla + \nabla^L\right)\right)^* = \frac{1}{2}\left(\nabla^* + \nabla^\dagger\right).$$

Since (para-)-holomorphicity of both ∇, ∇^L leads to (para-)-holomorphicity of $\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^L)$, we have established that $\tilde{\nabla}$ is a (para-)Chern connection.

□

Because (para-)holomorphicity of a connection ∇ is a weaker condition than the torsion coupling of ∇ with L, we have the following theorem as a slight modification of Theorem 12 .

Theorem 13. *Let* $(\mathfrak{M}, g, L)$ *be an almost (para-)Hermitian manifold with affine connections* ∇, ∇^*, ∇^L, *and* $\nabla^\dagger$. *If any one of them is torsion-coupled to* L *and any one of them is Codazzi-coupled to* g, *then all four connections are torsion-coupled to* L *and Codazzi-coupled to* g. *In this case,* L *must be integrable, and* $\mathfrak{M}$ *must be (para-)Hermitian.*

Theorem 13 was established in a preliminary report [4]. As in the case of Theorem 12, $\tilde{\nabla}$, when restricted to the bundle $T^{(1,0)}\mathfrak{M}$, must be equal to the (para-)Chern connection, due to $\tilde{\nabla}$ being (para-)holomorphic, (para-)complex, and (para-)Hermitian. In the theory of holomorphic vector bundles, the Chern connection is the unique Hermitian holomorphic connection on a holomorphic vector bundle, and in particular on $T^{(1,0)}\mathfrak{M}$ on complex manifolds [9]. In general, the (para-)Chern connection has torsion; however it is torsion-free on $T^{(1,0)}\mathfrak{M}$ if and only if the pair (g, L) define a (para-)Kähler structure.

3.6. Codazzi-(Para-)Hermitian Manifold

In the current investigation, connections are allowed to carry torsion. But when the (∇, L) pair is torsion-coupled (instead of Codazzi-coupled), L is still integrable. This leads to the definition of a *Codazzi-(para-)Hermitian manifold*.

Definition 14. An almost Codazzi-(para-)Hermitian manifold $(\mathfrak{M}, g, L, \nabla)$ is an almost (para-)Hermitian manifold $(\mathfrak{M}, g, L)$ with an affine connection ∇ (not necessarily torsion-free) which is Codazzi-coupled to g. When ∇ is further torsion-coupled to L, then L becomes integrable, so we will call $(\mathfrak{M}, g, L, \nabla)$ a Codazzi-(para-)Hermitian manifold instead.

So an almost Codazzi-(para-)Hermitian manifold is a Riemannian manifold $(\mathfrak{M}, g)$ in which g is compatible with both L in terms of (para-)Hermitianity, and with ∇ in terms of Codazzi coupling. It is an almost (para-)Hermitian manifold with an affine connection that is Codazzi-coupled to g. When L is further assumed to be torsion-coupled to ∇, then integrability of L is achieved, which turns $\mathfrak{M}$ into a (para-)Hermitian manifold $(\mathfrak{M}, g, L)$. Given an almost (para-)Hermitian manifold, such ∇ always exists, as one can take ∇ to be any (para-)Hermitian connection [3,7] that satisfies

$$\nabla g = 0 \text{ and } \nabla L = 0.$$

Similarly, any (para-)Hermitian manifold is trivially Codazzi-(para-)Hermitian, because one can always take Chern connection to be the desired ∇, with

$\theta(X, Y) = 0$ and $C(X, Y, Z) = 0$. In general, Codazzi-(para-)Hermitian manifolds have a quadruple of conjugate connections, $\nabla, \nabla^*, \nabla^L, \nabla^\dagger$, all possibly carrying torsion.

Previously, in [1] the following notion was introduced

Definition 15. An almost Codazzi-(para-)Kähler manifold $(\mathfrak{M}, g, L, \nabla)$ is an almost (para-)Hermitian manifold $(\mathfrak{M}, g, L)$ with an affine connection ∇ (not necessarily torsion-free) which is Codazzi-coupled to both g and L. If ∇ is torsion-free, then L is automatically integrable and ω is parallel (and hence d-closed), so in this case we will call $(\mathfrak{M}, g, L, \nabla)$ a Codazzi-(para-)Kähler manifold.

So, compared to almost Codazzi-(para-)Hermitian manifold, an almost Codazzi-(para-)Kähler manifold further requires (∇, L) to be Codazzi-coupled. An almost Codazzi-(para-)Hermitian manifold becomes a Codazzi-(para-)Hermitian manifold when (∇, L) is torsion-coupled, whereas an almost Codazzi-(para-)Kähler manifold becomes Codazzi-(para-)Kähler manifold when the torsion of ∇ vanishes.

Note that Codazzi coupling of (∇, L) and torsion coupling of (∇, L) are generally quite different compatibility conditions, with the former requiring the two torsions T^∇ and T^{∇^L} to equal and the latter causing them to obey a "twisting" relation (7).

In the case of Codazzi-(para-)Hermitian manifold, it may admit (Theorem 12) four (para-)holomorphic connections, $\nabla, \nabla^*, \nabla^L, \nabla^\dagger$, each of which are Codazzi coupled to g and torsion-coupled to L. In this scenario,

$$T^\nabla = T^{\nabla^*} \neq T^{\nabla^L} = T^{\nabla^\dagger}.$$

In the case of almost Codazzi-(para-)Kähler manifold, because ∇ is required to be Codazzi-coupled both to g and to L,

$$T^\nabla = T^{\nabla^L} = T^{\nabla^*} \neq T^{\nabla^\dagger}.$$

However, ∇^L may not be Codazzi-coupled to g; ∇^* may not be Codazzi-coupled to L. It is easy to show that ∇^L is Codazzi-coupled to g if and only if ∇^* is Codazzi-coupled to L . In this condition, $T^\nabla = T^{\nabla^\dagger}$, so $\nabla = \nabla^\dagger$, which leads to $\nabla^L = \nabla^*$.

Note that for torsion-free connections, torsion coupling of (∇, L) is equivalent to the Codazzi coupling of (∇, L). So a Codazzi-(para-)Hermitian manifold becomes a Codazzi-(para-)Kähler manifold, where the quadruple of conjugate connections for the former case reduce to a pair of torsion-free conjugate connections $\nabla = \nabla^\dagger$ and $\nabla^L = \nabla^*$ for the latter case. This was the case studied at length in [1,15].

From Theorem 12, $\tilde{\nabla} = \frac{1}{2}\left(\nabla + \nabla^L\right) = \frac{1}{2}\left(\nabla^* + \nabla^\dagger\right)$ satisfies $\tilde{\nabla}\omega = 0$. However, because

$$\begin{aligned} T^{\tilde{\nabla}}(X,Y) &= \frac{1}{2}\left(T^{\nabla}(X,Y) + T^{\nabla^L}(X,Y)\right) \\ &= \frac{1}{2}\left(T^{\nabla}(X,Y) + L^{-1}T^{\nabla}(LX,Y)\right) \neq 0, \end{aligned}$$

so in general $d\omega \neq 0$, which is the obstruction for an almost Codazzi-(para-)Kähler $\mathfrak{M}$ to become (para-)Kähler, unless $T^{\nabla} = T^{\nabla^L} = 0$. A Codazzi-(para-)Hermitian manifold remains (para-)Hermitian but not (para-)Kähler unless $T^{\nabla} = 0$.

To summarize, starting from an almost Codazzi-(para-)Hermitian manifold, which is an almost (para-)Hermitian manifold along with a connection ∇ that is Codazzi-coupled to g, we can generate either a Codazzi-(para-)Hermitian manifold (when the said ∇ is torsion-coupled to L) or a Codazzi-(para-)Kähler manifold (when the said ∇ is Codazzi-coupled to L). Further assuming $T^{\nabla} = 0$ in either of the above two cases will lead to a Codazzi-(para-)Kähler manifold.

So a question remains as whether a (para-)Kähler manifold may admit affine connections that carry torsion—this will happen if ∇ is torsion-coupled to L (and hence leading to integrability of L) and at the same time coupled with ω in some way such that $d\omega = 0$. This is the topic of the next section.

4. Interaction of ∇ with Almost (Para-)Kähler Structures (ω, L)

4.1. ω-Conjugate Connections

Let ω be a non-degenerate two-form on $\mathfrak{M}$, called an *almost symplectic structure*. Following [1], let us define the ω-conjugate transformation $\nabla^\dagger$ of ∇ by

$$Z\omega(X,Y) = \omega(\nabla^\dagger_Z X, Y) + \omega(X, \nabla_Z Y) = \omega(\nabla_Z X, Y) + \omega(X, \nabla^\dagger_Z Y) \quad (45)$$

where conjugation is invariantly defined with respect to *either* the first *or* the second entry of ω despite of the skew-symmetric nature of ω. As in the study of g-conjugation, we introduce

$$\Gamma(X,Y,Z) := (\nabla_Z \omega)(X,Y) = Z(\omega(X,Y)) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y),$$

which satisfies

$$\begin{aligned} \Gamma(X,Y,Z) &= \omega((\nabla^\dagger - \nabla)_Z X, Y) = -(\nabla^\dagger_Z \omega)(X,Y); \\ \Gamma(X,Y,Z) - \Gamma(Z,Y,X) &= \omega(T^{\nabla^\dagger}(Z,X) - T^{\nabla}(Z,X), Y). \end{aligned} \quad (46)$$

Because of the skew-symmetry $\Gamma(X,Y,Z) = -\Gamma(Y,X,Z)$, further requiring Codazzi coupling of (∇, ω), that is $\Gamma(X,Y,Z) = \Gamma(Z,Y,X)$, leads to $\Gamma(X,Y,Z) = 0$, that is, $(\nabla_Z \omega)(X,Y) = 0$.

Thus, we have the following:

Proposition 11. ([1]), Proposition 2.12 *Let $\nabla^\dagger$ denote ω-conjugate of an arbitrary connection ∇. The following are equivalent:*

(a) $\nabla\omega = 0$.
(b) $\nabla^\dagger\omega = 0$.
(c) $\nabla = \nabla^\dagger$.
(d) $T^\nabla = T^{\nabla^\dagger}$.

On an almost (para-)Hermitian manifold $(\mathfrak{M}, \omega, L)$, the non-degenerate skew-symmetric form ω and L satisfy the compatibility condition:

$$\omega(LX, Y) + \omega(X, LY) = 0$$

so that $g(X, Y) = \omega\left(L^{-1}X, Y\right)$ is now a compatible metric (assuming a taming condition holds $\omega(L^{-1}X, X) > 0$). Moreover, in this case, we know that $\nabla^\dagger = (\nabla^*)^L = \left(\nabla^L\right)^*$, see Proposition 10. The manifold $\mathfrak{M}$ becomes (para-)Kähler when $d\omega = 0$ and L is integrable. When writing out $d\omega$ with an arbitrary affine connection ∇, we have

Lemma 16. (Cartan's Lemma) *Let $(\mathfrak{M}, \omega)$ be an almost symplectic manifold with a non-degenerate skew-symmetric form. Then, for any vector fields X, Y, Z,*

$$\begin{aligned} d\omega(X, Y, Z) = {} & (\nabla_Z\omega)(X, Y) + (\nabla_X\omega)(Y, Z) + (\nabla_Y\omega)(Z, X) \\ & + \omega\left(T^\nabla(X, Y), Z\right) + \omega\left(T^\nabla(Y, Z), X\right) \\ & + \omega\left(T^\nabla(Z, X), Y\right). \end{aligned} \tag{47}$$

Of course, Cartan's Lemma holds for any affine connection, including $\nabla^\dagger$, which reads

$$\begin{aligned} d\omega(X, Y, Z) = {} & \left(\nabla^\dagger_Z\omega\right)(X, Y) + \left(\nabla^\dagger_X\omega\right)(Y, Z) + \left(\nabla^\dagger_Y\omega\right)(Z, X) \\ & + \omega\left(T^{\nabla^\dagger}(X, Y), Z\right) + \omega\left(T^{\nabla^\dagger}(Y, Z), X\right) \\ & + \omega\left(T^{\nabla^\dagger}(Z, X), Y\right). \end{aligned} \tag{48}$$

4.2. Coupling of (∇, ω) that Leads to $d\omega = 0$

In this section we will consider couplings of connections with an almost symmetric structure ω on $\mathfrak{M}$ that lead to ω being d-closed.

Lemma 17. *Given ω an arbitrary ∇ with torsion T^∇ and ω. Then, any two of the following three statements lead to the third:*

(a) $\omega(T^\nabla(X, Y), Z) + (\nabla_Z\omega)(X, Y) = 0$.
(b) $\omega(T^\nabla(X, Y), Z) - (\nabla_Z\omega)(X, Y) + (\nabla_X\omega)(Y, Z) - (\nabla_Y\omega)(X, Z) = 0$.
(c) $\omega(T^\nabla(X, Y), Z) + \frac{1}{2}((\nabla_X\omega)(Y, Z) - (\nabla_Y\omega)(X, Z)) = 0$.

Proof. Direct substitutions. $\square$

We note that, using (46), case (b) can be written in an analogous form of case (a)

$$\omega(T^{\nabla^\dagger}(X,Y),Z) + (\nabla_Z \omega^\dagger)(X,Y) = 0,$$

whereas case (c) can be written as

$$T^{\nabla} = -T^{\nabla^\dagger}.$$

Proposition 12. *Given an almost symplectic manifold $(\mathfrak{M},\omega)$ with an arbitrary affine connection ∇, then ω is d-closed, i.e., $d\omega = 0$, if any of following three torsion conditions* (49)–(51) *holds:*

$$\omega(T^{\nabla}(X,Y),Z) + (\nabla_Z \omega)(X,Y) = 0; \tag{49}$$

$$\omega(T^{\nabla^\dagger}(X,Y),Z) + (\nabla^\dagger_Z \omega)(X,Y) = 0; \tag{50}$$

$$T^{\nabla} = -T^{\nabla^\dagger}. \tag{51}$$

From Lemma 17, any two of the three equations (49), (50), and (51) lead to the third to hold; these three conditions are alternative expressions of statements (a)–(c) of Lemma 17.

Proof. Perform permutation of X, Y, Z and then direct substitute either (49) or (b) into the right-hand side of (47), or (50) into the right-hand side of (48), leads to 0. Hence in all three cases, $d\omega = 0$. □

On an almost (para-)Kähler manifold $(\mathfrak{M},\omega,L)$, either (50) or (51) guaranteeing $d\omega = 0$, in conjunction with torsion coupling of (∇, L) that guarantees $N_L = 0$, will lead $\mathfrak{M}$ to become (para-)Kähler. In such cases, the coupling of g and ∇^L is no longer Codazzi, as we will see below.

Proposition 13. *Suppose $(\mathfrak{M},\omega,L)$ is almost (para-)Kähler with an affine connection ∇ that is torsion-coupled to L and coupled with ω through either* (49) *or* (51).

(i) *If (∇,ω) satisfies* (49)*, then*

$$T^{\nabla}(Y,Z) = T^{\nabla^\dagger}(Y,Z) + \left(T^{\nabla^L}\right)^t(Y,Z) - \left(T^{\nabla^L}\right)^t(Z,Y) \tag{52}$$

where $\left(T^{\nabla^L}\right)^t(\cdot,Z)$ is the transpose (adjoint) of the endomorphism $T^{\nabla}(\cdot,Z)$ with respect to g, and equivalently, the coupling of g with ∇^L is given by

$$\begin{aligned}(\nabla^L_X g)(Y,Z) - (\nabla^L_Y g)(X,Z) &= g\left(T^{\nabla}(X,Y),Z\right) - g\left(T^{\nabla^L}(X,Y),Z\right) - g\left(T^{\nabla^L}(Z,Y),X\right)\\ &\quad + g\left(T^{\nabla^L}(Z,X),Y\right).\end{aligned} \tag{53}$$

(ii) *If* (∇, ω) *satisfies* (51), *then*

$$T^{\nabla}(Y,Z) = -T^{\nabla^{\dagger}}(Y,Z),$$

and the coupling of g *with* ∇^L *is given by*

$$\left(\nabla^L_X g\right)(Y,Z) - \left(\nabla^L_Y g\right)(X,Z) = -g\left(T^{\nabla}\left(X,Y\right),Z\right) - g\left(T^{\nabla^L}(X,Y),Z\right). \quad (54)$$

Proof. Suppose (g, ω, L) is a compatible triple and the pair (∇, L) is torsion-coupled.

(i) Suppose ω satisfies (49). Then, (46) implies

$$-\omega\left(T^{\nabla}\left(Y,Z\right),X\right) + \omega\left(T^{\nabla}\left(X,Z\right),Y\right) = \omega\left(T^{\nabla^{\dagger}}(X,Y),Z\right) - \omega\left(T^{\nabla}\left(X,Y\right),Z\right), \quad (55)$$

which, in terms of g may be rewritten as

$$g\left(T^{\nabla}\left(Y,Z\right),LX\right) - g\left(T^{\nabla}\left(X,Z\right),LY\right) = g\left(LT^{\nabla^{\dagger}}(X,Y),Z\right) - g\left(LT^{\nabla}\left(X,Y\right),Z\right), \quad (56)$$

and therefore,

$$-\left(T^{\nabla}\right)^t\left(LX,Y\right) + \left(T^{\nabla}\right)^t\left(LY,X\right) = LT^{\nabla^{\dagger}}\left(X,Y\right) - LT^{\nabla}\left(X,Y\right) \quad (57)$$

where $\left(T^{\nabla}\right)^t$ is the adjoint of T^{∇} defined by

$$g\left(T^{\nabla}\left(X,Y\right),Z\right) = g\left(X,\left(T^{\nabla}\right)^t\left(Z,Y\right)\right).$$

Since (∇, L) is torsion-coupled, we have

$$T^{\nabla}\left(LX,Y\right) = LT^{\nabla^L}\left(X,Y\right). \quad (58)$$

From this,

$$\begin{aligned} g\left(X, L^{-1}\left(T^{\nabla}\right)^t\left(LY,Z\right)\right) &= -g\left(T^{\nabla}\left(L^{-1}X,Z\right),LY\right) \\ &= g\left(L^{-1}T^{\nabla}\left(LX,Z\right),Y\right) \\ &= g\left(T^{\nabla^L}\left(X,Z\right),Y\right) \\ &= g\left(X,\left(T^{\nabla^L}\right)^t\left(Y,Z\right)\right) \end{aligned}$$

and therefore,

$$L^{-1}\left(T^{\nabla}\right)^t\left(LY,Z\right) = \left(T^{\nabla^L}\right)^t\left(Y,Z\right), \quad (59)$$

so that (57) gives (52). Using the relation (31) for ∇^L and $\nabla^{\dagger} = \left(\nabla^L\right)^*$, we obtain the following identity

$$\left(\nabla^L_X g\right)(Y,Z) - \left(\nabla^L_Y g\right)(X,Z) = g\left(T^{\nabla^{\dagger}}\left(X,Y\right),Z\right) - g\left(T^{\nabla^L}(X,Y),Z\right), \quad (60)$$

from which we see that (52) is equivalent to (53).

(ii) Suppose that instead, (∇, ω) satisfy (51). Applying this to the identity (60), which is always true, leads to (54).

□

Note that $\left(T^{\nabla^L}\right)^t$ is skew-symmetric if and only if T^{∇^L} is a totally skew-symmetric torsion, which is not always the case.

From Theorem 11 we know that if the pair (∇, L) is torsion-coupled, (∇, g) is Codazzi-coupled if and only if so is $\left(\nabla^L, g\right)$. We have now seen that in both cases (49) and (51), the pair $\left(\nabla^L, g\right)$ is generally not Codazzi-coupled, and hence neither is (∇, g). However, in both cases $d\omega = 0$ and assuming torsion coupling of (∇, L), L is integrable, which leads to (para-)Kähler structures.

These are the cases of (para-)Kähler manifolds admitting four (!) (para-)holomorphic connections $\nabla, \nabla^*, \nabla^L, \nabla^\dagger$ all possibly carrying non-vanishing torsions:

$$T^\nabla = -T^{\nabla^\dagger} \neq T^{\nabla^L} = -T^{\nabla^*}.$$

However, these are not Codazzi-(para-)Kähler manifolds.

5. Summary and Discussions

The Codazzi coupling of an affine connection ∇ with a Riemannian metric g is a well-studied concept in affine geometry [10,11]. The robustness of the Codazzi coupling was investigated by perturbing both the metric and the affine connection [12] and by its interaction with other transformations of connections [13]. Codazzi coupling of ∇ with g forms the foundation of the interdisciplinary field of information geometry—torsion-freeness of ∇ is a defining characteristic of what is known as "statistical structure" in information geometry. It was recently shown [1] that a statistical structure can be enhanced to a (para-)Kä hler structure when the manifold is equipped with an almost (para-)complex operator L that is Codazzi-coupled to a torsion-free ∇.

The situation with connections admitting torsion, however, was far less clear, though it may arise in practice from the so-called pre-contrast functions [5] in information geometry. To allow for torsion, in this paper, we relax the Codazzi coupling condition of (∇, L) to a torsion coupling condition (7), and show that torsion coupling of L with ∇ leads to integrability of L *and* (para-)holomorphicity of ∇ and of ∇^L. On the other hand, we also find that Codazzi coupling of g with a (para-)holomorphic ∇ would lead ∇^*, the g-conjugate connection, to be (para-)holomorphic. These findings lead to Codazzi-(para-)Hermitian manifold as a relaxation of Codazzi-(para-)-Kähler in terms of assumed torsion properties. In the former case with torsion-admitting connections, $d\omega \neq 0$. An equivalent way of conceiving Codazzi-(para-)Hermitian manifolds is that it is a (para-)Hermitian manifold with four, possibly torsion-carrying connections $\nabla, \nabla^L, \nabla^*$, and $\nabla^\dagger \equiv \nabla^{*L} \equiv \nabla^{L*}$, that are all (i) (para-)holomorphic with respect to L; (ii) Codazzi-coupled to g. When torsion of any connection is zero (and hence torsion of all four connections are zero), the Codazzi-(para-)Hermitian manifold becomes a Codazzi-(para-)Kähler manifold; in this case, $\nabla^L = \nabla^*$, so there are only a pair of torsion-free (para-)holomorphic connections.

(Para-)holomorphic connections have hardly been systematically studied in information geometry except in the restricted setting of flat connections (see [2]). Connections investigated in this paper are generally neither curvature-free nor torsion-free; they are generally neither parallel with respect to g nor to L. It is remarkable that when (∇, L) is torsion-coupled and (∇, g) is Codazzi-coupled, then an integrable structure, i.e., a (para-)Hermitian manifold results. Even (para-)Kähler manifolds may admit quadruples of torsion-carrying connections though these will not be Codazzi-(para-)Kähler manifolds. So our investigation opens the door for studying integrable structures in the presence of connections with torsion.

References

[1] Fei, T., Zhang, J.: Interaction of Codazzi couplings with (para-)Kähler geometry. Results Math. **72**, 2037–2056 (2017)

[2] Furuhata, H.: Hypersurfaces in statistical manifolds. Diff. Geom. Appl. **27**(3), 420–429 (2009)

[3] Gauduchon, P.: Hermitian connections and Dirac operators Boll. Un. Mat. Ital. B **7**(11), 257–288 (1997)

[4] Grigorian, S., Zhang, J.: (Para-)holomorphic connections for information geometry. In: International Conference on Geometric Science of Information (GSI 2017), pp. 186–194, Springer, Cham (2017)

[5] Henmi, M., Matsuzoe, H.: Geometry of pre-contrast functions and non–conservative estimating functions. In: International Workshop on Complex Structures, Integrability and Vector Fields, Vol. 1340, No. 1. AIP Publishing (2011)

[6] Hitchin, N.: The moduli space of complex Lagrangian submanifolds. In: Yau, S.T. (ed.) Surveys in Differential Geometry VII, pp. 327–345. International Press (2000)

[7] Ivanov, S., Zamkovoy, S.: Parahermitian and paraquaternionic manifolds. Diff. Geom. Appl. **23**(2), 205–234 (2005)

[8] Lauritzen, S.L.: Statistical manifolds. In: Differential Geometry in Statistical Inference. IMS Lecture Notes Monograph Series, vol. 10, pp. 163–216. Institute of Mathematical Statistics (1987)

[9] Moroianu, A.: Lectures on Kähler Geometry, vol. 69. London Mathematical Society Student Texts. Cambridge University Press, Cambridge (2007)

[10] Nomizu, K., Sasaki, T.: Affine Differential Geometry: Geometry of Affine Immersions, vol. 111. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1994)

[11] Simon, U.: Affine differential geometry. In: Dillen, F.J., Verstraelen, L.C. (eds.) Handbook of Differential Geometry, vol. 1, pp. 905–961. North-Holland, (2000)

[12] Schwenk-Schellschmidt, A., Simon, U.: Codazzi-equivalent affine connections. Results Math. **56**(1–4), 211–229 (2009)

[13] Tao, J., Zhang, J.: Transformations and coupling relations for affine connections. Diff. Geom. Appl. **49**, 111–130 (2016)

[14] Yang, B., Zheng, F.: On compact Hermitian manifolds with flat Gauduchon connections. Acta Math. Sin. Engl. Ser. **34**, 1259–1268 (2018)

[15] Zhang, J., Teng, F.: Information geometry with (para-)Kähler structures. In: Information Geometry and its Applications IV, pp. 297-321. Springer Proceedings in Mathematics and Statistics (PROMS) series, vol. 252 (2018)

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