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Book Title	Information Geometry and Its Applications
Series Title	
Chapter Title	Information Geometry with (Para-)Kähler Structures
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Abstract

We investigate conditions under which a statistical manifold $\mathfrak{M}$ (with a Riemannian metric g and a pair of torsion-free conjugate connections ∇, ∇^*) can be enhanced to a (para-)Kähler structure. Assuming there exists an almost (para-)complex structure L compatible with g on a statistical manifold $\mathfrak{M}$ (of even dimension), then we show $(\mathfrak{M}, g, L, \nabla)$ is (para-)Kähler if ∇ and L are Codazzi coupled. Other equivalent characterizations involve a symplectic form $\omega \equiv g(L\cdot, \cdot)$. In terms of the compatible triple (g, ω, L), we show that (i) each object in the triple induces a conjugate transformation on ∇ and becomes an element of an (Abelian) Klein group; (ii) the compatibility of any two objects in the triple with ∇ leads to the compatible quadruple (g, ω, L, ∇) in which any pair of objects are mutually compatible. This is what we call *Codazzi-(para-)Kähler manifold* [8] which admits the family of torsion-free α-connections (convex mixture of ∇, ∇^*) compatible with (g, ω, L). Finally, we discuss the properties of divergence functions on $\mathfrak{M} \times \mathfrak{M}$ that lead to Kähler (when $L = J, J^2 = -id$) and para-Kähler (when $L = K, K^2 = id$) structures.

Keywords (separated by '-'): Statistical manifold - Torsion - Codazzi coupling - Conjugation of connection - Kähler structure - Para-Kähler structure - Codazzi-(para-)Kähler - Compatible triple - Compatible quadruple

Information Geometry with (Para-)Kähler Structures

Jun Zhang and Teng Fei

Abstract We investigate conditions under which a statistical manifold $\mathfrak{M}$ (with a Riemannian metric g and a pair of torsion-free conjugate connections ∇, ∇^*) can be enhanced to a (para-)Kähler structure. Assuming there exists an almost (para-)complex structure L compatible with g on a statistical manifold $\mathfrak{M}$ (of even dimension), then we show $(\mathfrak{M}, g, L, \nabla)$ is (para-)Kähler if ∇ and L are Codazzi coupled. Other equivalent characterizations involve a symplectic form $\omega \equiv g(L\cdot, \cdot)$. In terms of the compatible triple (g, ω, L), we show that (i) each object in the triple induces a conjugate transformation on ∇ and becomes an element of an (Abelian) Klein group; (ii) the compatibility of any two objects in the triple with ∇ leads to the compatible quadruple (g, ω, L, ∇) in which any pair of objects are mutually compatible. This is what we call *Codazzi-(para-)Kähler manifold* [8] which admits the family of torsion-free α-connections (convex mixture of ∇, ∇^*) compatible with (g, ω, L). Finally, we discuss the properties of divergence functions on $\mathfrak{M} \times \mathfrak{M}$ that lead to Kähler (when $L = J, J^2 = -id$) and para-Kähler (when $L = K, K^2 = id$) structures.

Keywords Statistical manifold · Torsion · Codazzi coupling · Conjugation of connection · Kähler structure · Para-Kähler structure · Codazzi-(para-)Kähler · Compatible triple · Compatible quadruple

1 Introduction

Let $\mathfrak{M}$ be a smooth (real) manifold *of even dimension* and ∇ be an affine connection on it. In this paper, we would investigate the interaction of ∇ with three geometric structures on $\mathfrak{M}$, namely, a pseudo-Riemannian metric g, a nondegenerate 2-form

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N. Ay et al. (eds.), *Information Geometry and Its Applications*,
Springer Proceedings in Mathematics & Statistics 252,
https://doi.org/10.1007/978-3-319-97798-0_11

ω, and a tangent bundle isomorphism $L : T\mathfrak{M} \to T\mathfrak{M}$, often forming a "compatible triple" together. The interaction of the compatible triple (g, ω, L) with ∇, in terms of parallelism, is well understood, leading to integrability of L and of ω, and turning almost (para-)Hermitian structure of $\mathfrak{M}$ to (para-)Kähler structure on $\mathfrak{M}$. Here, we investigate the interaction of ∇ with the compatible triple (g, ω, L) in terms of Codazzi coupling, a relaxation of parallelism.

We start by recalling that the statistical structure $(\mathfrak{M}, g, \nabla)$ can be defined either as (i) a manifold $(\mathfrak{M}, g, \nabla, \nabla^*)$ with a pair, ∇ and ∇^*, of torsion-free g-conjugate connections (Lauritzen's definition [21]); or (ii) a manifold $(\mathfrak{M}, g, \nabla)$ with a torsion-free connection ∇ that is Codazzi coupled to g (Kurose's definition [20]). Though the two definitions can be shown to be equivalent, they represent two different perspectives of generalizing Levi-Civita connection which is, by definition, parallel to g. Section 2.1 aims at clarifying the distinction and the link between (i) the concept of h-conjugate transformation of connection ∇; and (ii) the concept of Codazzi coupling associated to the pair (∇, h), where h is an arbitrary (0, 2)-tensor. The special cases of $h = g$ (symmetric) and $h = \omega$ (skew-symmetric) are highlighted, because both g-conjugation and ω-conjugation are involutive operations. Codazzi coupling ∇ with h then, is the precise characterization of the condition for such conjugate operations on a connection to preserve its torsion.

In Sect. 2.2, we investigate Codazzi coupling of ∇ with a tangent bundle isomorphism L, in particular the cases of $L = J, J^2 = -id$ (almost complex structure) and $L = K, K^2 = id$ (almost para-complex structure, with same multiplicity for ± 1 eigenvalues). Such coupling is shown to lead to integrability of L.

In Sects. 2.3 and 2.4, the interaction of ∇ with the compatible triple (g, ω, L) is studied. We follow the same approach of Sect. 2.1 in distinguishing (i) the conjugation transformation of ∇ by, and (ii) the Codazzi coupling of ∇ with respect to each of the (g, ω, L). In the former case (Sect. 2.3), we show that g-conjugate, ω-conjugate, and L-gauge transformation (together with identity transform) form a Klein group of transformations of connections. In the latter case (Sect. 2.4), we show that Codazzi couplings of ∇ with any two of the compatible (g, ω, L) lead to its coupling with the third (and hence turning the compatible triple into a compatible quadruple (g, ω, L, ∇)). After studying the implications of the existence of such couplings (Sect. 2.5), this then leads to the definition of *Codazzi-(para-)Kähler* structure (Sect. 2.6); its relations with various other geometric structures (Hermitian, symplectic, etc) are also discussed there.

Section 3 investigates how (para-)Kähler structures on $\mathfrak{M} \times \mathfrak{M}$ may arise from divergence functions. After a brief review how divergence functions induce a statistical structure (Sect. 3.1), we study how they may induce a symplectic structure on $\mathfrak{M} \times \mathfrak{M}$ (Sect. 3.2). We then show constraints on divergence functions if the induced structures on $\mathfrak{M} \times \mathfrak{M}$ are further para-Kähler (Sect. 3.3) or Kähler (Sect. 3.4). As an exercise, we relate our construction of Kähler structure to Calabi's diastatic function approach (Sect. 3.5).

In this paper, we investigate integrability of L and of ω while g and L are not necessarily covariantly constant (i.e., parallel) with respect to ∇, but instead are Codazzi coupled to it. The results were known in the parallel case: the exis-

440419_1_En_11_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

tence of a torsion-free connection ∇ on $\mathfrak{M}$ such that $\nabla g = 0$ (metric-compatible) and $\nabla L = 0$ (complex connection) implies that $(\mathfrak{M}, g, L)$ is (para-)Kähler. When Codazzi coupling replaces parallelism, our results show that (para-)Kähler manifolds may still admit a *pair* of conjugate connections ∇ and ∇^*, much like statistical manifolds do. In recent work [15], we showed that such pair of connections are in fact both (para-)holomorphic for the (para-)Kähler manifolds; general conditions for (para-)holomorphicity of g-conjugate and L-gauge transformations of connections for (para-)Hermitian manifolds are also studied there.

As most materials in Sect. 2 has already appeared in [8, 31], we only provide summary of results while omitting proofs. A small improvement to earlier results is showing the entire family of α-connections for the Codazzi-(para-)Kähler manifold. Section 3 contains results unpublished before. All materials of this paper were first presented at the fourth international conference on Information Geometry and Its Applications (IGAIA4).

2 Enhancing Statistical Structure to (Para-)Kähler Structures

In this Section, we investigate Codazzi couplings of an affine connection ∇ on a real manifold $\mathfrak{M}$ with a pseudo-Riemannian metric g, a symplectic form ω, and a tangent bundle isomorphism $L : T\mathfrak{M} \to T\mathfrak{M}$. We prove that the Codazzi coupling between a torsion-free ∇ and a quadratic operator L leads to transversal foliations, and that the Codazzi coupling of any two of (g, ω, L) leads to the Codazzi coupling of the remaining third. Mirroring the study of these Codazzi couplings is the study of the transformations of ∇ by g-conjugation, by ω-conjugation, and by L-gauge, and of how their torsions are affected including when they are preserved. As a highlight, we show that these transformations generically are non-trivial elements of a four-element Klein group. This motivates the notions of *compatible quadruple* and *Codazzi-(para-) Kähler* manifolds.

2.1 Conjugate Transformation and Codazzi Coupling Associated to (h, ∇)

The simplest form of "coupling" relation between ∇ and h is that of "parallelism": $\nabla h = 0$. In other words, covariant derivative of h under ∇ is zero. There are two ways of generalizing this notion of parallelism: the first involves introducing the notion of a h-conjugate transformation ∇^h of ∇ such that $\nabla^h = \nabla$ recovers $\nabla h = 0$, the second involves requiring ∇h to have some symmetry for which $\nabla h = 0$ is a special case. Below, we discuss them in turn.

Conjugation of a connection by h If h is any non-degenerate $(0, 2)$-tensor, i.e., bilinear form, it induces isomorphisms $h(X, -)$ and $h(-, X)$ from vector fields X to one-forms. When h is not symmetric, these two isomorphisms are different. Given an affine connection ∇, we can take the covariant derivative of the one-form $h(X,-)$ with respect to Z, and obtain a non-tensorial object θ such that, when fixing X,

$$\theta_Z(Y) = Z(h(X,Y)) - h(X, \nabla_Z Y).$$

Similarly, we can take the covariant derivative of the one-form $h(-,Y)$ with respect to Z, and obtain a corresponding object $\tilde{\theta}$ such that, when fixing Y,

$$\tilde{\theta}_Z(X) = Z(h(X,Y)) - h(\nabla_Z X, Y).$$

Since h is non-degenerate, there exists a U and V such that $\theta_Z = h(U,-)$ and $\tilde{\theta}_Z = h(-,V)$ as one-forms, so that

$$Z(h(X,Y)) = h(U(Z,X),Y) + h(X, \nabla_Z Y),$$
$$Z(h(X,Y)) = h(\nabla_Z X, Y) + h(X, V(Z,Y)).$$

Proposition 1 ([31], Proposition 7) *Let* $\nabla_Z^{\texttt{left}} X := U(Z,X)$ *and* $\nabla_Z^{\texttt{right}} X := V(Z,X)$. *Then* $\nabla^{\texttt{left}}$ *and* $\nabla^{\texttt{right}}$ *are both affine connections as induced from* ∇.

The $\nabla^{\texttt{left}}$ and $\nabla^{\texttt{right}}$ are called, respectively, *left-h-conjugate* and *right-h-conjugate* of ∇; neither is involutive in general. From their definitions, it is easy to see that

$$(\nabla^{\texttt{left}})^{\texttt{right}} = (\nabla^{\texttt{right}})^{\texttt{left}} = \nabla.$$

Reference [31] provided the conditions under which left- and right-conjugate of h are the same.

Proposition 2 ([31], Proposition 15) *When the non-degenerate bilinear form* h *is either symmetric,* $h(X,Y) = h(Y,X)$, *or skew-symmetric,* $h(X,Y) = -h(Y,X)$, *then*

$$\nabla^{\texttt{left}} = \nabla^{\texttt{right}}.$$

The two special cases of h: symmetric or skew-symmetric bilinear form, are denoted as g and ω, respectively. Since the left- and right-conjugates with respect to such h are equal, we use ∇^* to denote g-conjugate and $\nabla^\dagger$ to denote ω-conjugate of an arbitrary affine connection ∇. Note that both g-conjugation and ω-conjugation operations are involutive: $(\nabla^*)^* = \nabla$, $(\nabla^\dagger)^\dagger = \nabla$.

In information geometry, it is standard to consider α-connections for $\alpha \in \mathbb{R}$

$$\nabla_g^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*, \qquad \text{with} \quad (\nabla_g^{(\alpha)})^* = \nabla_g^{(-\alpha)}.$$

Likewise, we can introduce

440419_1_En_11_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

$$\nabla_\omega^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^\dagger, \qquad \text{with} \;\; (\nabla_\omega^{(\alpha)})^\dagger = \nabla_\omega^{(-\alpha)}.$$

Remark 1 Despite of the skew-symmetric nature of ω, ω-conjugation is one and the same whether defined with respect to the first or second slot of ω:

$$Z\omega(X,Y) = \omega(\nabla_Z^\dagger X, Y) + \omega(X, \nabla_Z Y) = \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z^\dagger Y).$$

Codazzi coupling of ∇ and h We introduce the (0,3)-tensor C defined by:

$$C_h(X,Y,Z) \equiv (\nabla_Z h)(X,Y) = Z(h(X,Y)) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y). \tag{1}$$

The tensor C_h is called the *cubic form* associated with (∇, h) pair. Rewriting the above

$$C_h(X,Y,Z) = h((\nabla^{\mathtt{left}} - \nabla)_Z X, Y) = h(X, (\nabla^{\mathtt{right}} - \nabla)_Z Y), \tag{2}$$

we see that

$$\nabla = \nabla^{\mathtt{left}} = \nabla^{\mathtt{right}}$$

if any only if $C_h = 0$. In this case, we say that ∇ is *parallel* to the bilinear form h, i.e.,

$$Z(h(X,Y)) = h(\nabla_Z X, Y) + h(X, \nabla_Z Y).$$

In general, the cubic forms associated with $(\nabla^{\mathtt{left}}, h)$ pair and with $(\nabla^{\mathtt{right}}, h)$ pair are

$$(\nabla_Z^{\mathtt{left}} h)(X,Y) = (\nabla_Z^{\mathtt{right}} h)(X,Y) = -C_h(X,Y,Z) = -(\nabla_Z h)(X,Y).$$

From (2), we can derive

$$\begin{aligned} C_h(X,Y,Z) - C_h(Z,Y,X) &= h(T^{\nabla^{\mathtt{left}}}(Z,X) - T^{\nabla}(Z,X), Y) \\ &= h(X, T^{\nabla^{\mathtt{right}}}(Z,Y) - T^{\nabla}(Z,Y)). \end{aligned}$$

So $C_h(X,Y,Z) = C_h(Z,Y,X)$ if and only if the torsions of ∇, $\nabla^{\mathtt{left}}$, $\nabla^{\mathtt{right}}$ are all equal

$$T(X,Y) = T^{\nabla^{\mathtt{left}}}(X,Y) = T^{\nabla^{\mathtt{right}}}(X,Y).$$

Definition 1 Let h be a non-degenerate bilinear form, and ∇ an affine connection. Then (∇, h) is called a *Codazzi pair*, and ∇ and h are said to be *Codazzi coupled*, if

$$C_h(X, Y, Z) = C_h(Z, Y, X) \tag{3}$$

or explicitly

$$(\nabla_Z h)(X, Y) = (\nabla_X h)(Z, Y).$$

Now, let us investigate Codazzi coupling of ∇ with g (symmetric case) or ω (skew-symmetric case); in both cases $\nabla^{\texttt{left}} = \nabla^{\texttt{right}}$.

- For $h = g$: symmetry of g implies $C_g(X, Y, Z) = C_g(Y, X, Z)$. This, combined with (3), leads to

  $$C_g(Z, Y, X) = C_g(X, Y, Z) = C_g(Y, X, Z) = C_g(Z, X, Y) = C_g(X, Z, Y) = C_g(Y, Z, X),$$

  so $C_g(X, Y, Z) \equiv \nabla g$ is totally symmetric in its three slots.
- For $h = \omega$: skew-symmetry of ω implies $C_\omega(X, Y, Z) = -C_\omega(Y, X, Z)$. This, combined with (3), leads to

  $$\begin{aligned} C_\omega(X, Y, Z) &= C_\omega(Z, Y, X) = -C_\omega(Y, Z, X) = -C_\omega(X, Z, Y) \\ &= C_\omega(Z, X, Y) = C_\omega(Y, X, Z) = -C_\omega(X, Y, Z), \end{aligned}$$

  hence $C_\omega(X, Y, Z) \equiv \nabla\omega = 0$.

We therefore conclude

Proposition 3 *Let ∇^* and $\nabla^\dagger$ denote the g-conjugate and ω-conjugate of an arbitrary connection ∇ with respect to g and ω, respectively.*

1. *The following are equivalent:*

   (i) (∇, g) *is Codazzi-coupled;*
   (ii) (∇^*, g) *is Codazzi-coupled;*
   (iii) ∇g *is totally symmetric;*
   (iv) $\nabla^* g$ *is totally symmetric;*
   (v) $T^\nabla = T^{\nabla^*}$.

2. *The following are equivalent:*

   (i) $\nabla\omega = 0$;
   (ii) $\nabla^\dagger\omega = 0$;
   (iii) $\nabla = \nabla^\dagger$;
   (iv) $T^\nabla = T^{\nabla^\dagger}$.

2.2 *Tangent Bundle Isomorphism*

Codazzi coupling of ∇ with L For a smooth manifold $\mathfrak{M}$, an isomorphism L of the tangent bundle $T\mathfrak{M}$ is a smooth section of the bundle End $T\mathfrak{M}$ such that it is invertible everywhere. Starting from a (not necessarily torsion-free) connection ∇ on $\mathfrak{M}$, an *L-gauge transformation* of a connection ∇ is a new connection ∇^L defined by

$$\nabla^L_X Y = L^{-1}(\nabla_X (LY))$$

for any vector fields X and Y. It can be verified that indeed ∇^L is an affine connection. Note that gauge transformations of a connection form a group, with operator composition as group multiplication.

Definition 2 L and ∇ are said to be *Codazzi coupled* if the following identity holds

$$(\nabla_X L)Y = (\nabla_Y L)X, \tag{4}$$

where

$$(\nabla_X L)Y \equiv \nabla_X (LY) - L(\nabla_X Y).$$

We have the following characterization of Codazzi coupling of ∇ with L

Lemma 1 (e.g., [27]) *Let ∇ be an affine connection, and let L be a tangent bundle isomorphism. Then the following statements are equivalent:*

(i) (∇, L) *is Codazzi-coupled.*
(ii) (∇^L, L^{-1}) *is Codazzi-coupled.*
(iii) $T^{\nabla} = T^{\nabla^L}$.

Integrablility of L A tangent bundle isomorphism $L : T\mathfrak{M} \to T\mathfrak{M}$ is said to be a *quadratic operator* if it satisfies a real coefficient quadratic polynomial equation with distinct roots, i.e., there exists $\alpha \neq \beta \in \mathbb{C}$ such that $\alpha + \beta,\ \alpha\beta$ are real numbers and

$$L^2 - (\alpha + \beta)L + \alpha\beta \cdot \mathrm{id} = 0.$$

Note that L is an isomorphism, so $\alpha\beta \neq 0$.

Let E_α and E_β be the eigenbundles of L corresponding to the eigenvalues α and β respectively, i.e., at each point $x \in \mathfrak{M}$, the fiber is defined by

$$E_\lambda(x) := \{X \in T_x\mathfrak{M} : L_x(X) = \lambda X\} \text{ for } \lambda = \alpha, \beta.$$

As subbundles of the tangent bundle $T\mathfrak{M}$, E_α and E_β are distributions. We call $E_\alpha(E_\beta)$ a foliation if for any vector fields X, Y with value in $E_\alpha(E_\beta)$, so is their Lie bracket $[X, Y]$.

The Nijenhuis tensor N_L associated with L is defined as

$$N_L(X, Y) = -L^2[X, Y] + L[X, LY] + L[LX, Y] - [LX, LY].$$

440419_1_En_11_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

When $N_L = 0$, the operator L is said to be integrable. In this case, the eigen-bundles of L form foliations, i.e., subbundles that are closed with respect to Lie bracket operation $[\cdot, \cdot]$.

One can derive (see [8]) that, when a quadratic operator L is Codazzi-coupled to an affine connection ∇, then the Nijenhuis tensor N_L has the expression

$$N_L(X, Y) = L^2 T^\nabla(X, Y) - LT^\nabla(X, LY) - LT^\nabla(LX, Y) + T^\nabla(LX, LY).$$

An immediate consequence is that $N_L = 0$ vanishes when ∇ is torsion-free $T^\nabla = 0$. That is,

Proposition 4 *A quadratic operator L is integrable if it is Codazzi-coupled to a torsion-free connection ∇.*

Combining Proposition 4 with Lemma 1 yields

Corollary 1 *A quadratic operator L is integrable if there exists a torsion-free connection ∇ such that ∇^L is torsion-free.*

Almost-complex J and almost-para-complex K operator The most important examples of the bundle isomorphism L are almost complex structures and almost para-complex structures. By definition, L is called an *almost complex structure* if $L^2 = -\mathrm{id}$. Analogously, L is known as an *almost para-complex structure* if $L^2 = \mathrm{id}$ and the multiplicities of the eigenvalues ± 1 are equal. We will use J and K to denote almost complex structures and almost para-complex structures, respectively, and use L when these two structures can be treated in a unified way. It is clear from our definitions that such structures exist only when $\mathfrak{M}$ is of even dimension.

The following results follow readily from Lemma 1 for the special case of $L^2 = \pm\mathrm{id}$.

Corollary 2 *When $L = J$ or $L = K$,*

1. $\nabla^L = \nabla^{L^{-1}}$, *i.e., L-gauge transformation is involutive,* $(\nabla^L)^L = \nabla$.
2. (∇, L) *is Codazzi-coupled if and only if* (∇^L, L) *is Codazzi-coupled.*

Compatible triple (g, ω, L) The compatibility condition between a metric g and an almost (para-)complex structure $J(K)$ is well-known, where $J^2 = -id$ and $K^2 = id$. We say that g is compatible with J if J is orthogonal, i.e.

$$g(JX, JY) = g(X, Y) \tag{5}$$

holds for any vector fields X and Y. Similarly we say that g is compatible with K if

$$g(KX, KY) = -g(X, Y) \tag{6}$$

is always satisfied, which implies that g must be of split signature. When expressed using L, (5) and (6) have the same form

$$g(X, LY) + g(LX, Y) = 0. \tag{7}$$

Hence a two-form ω can be defined

$$\omega(X, Y) = g(LX, Y), \tag{8}$$

and turns out to satisfy

$$\omega(X, LY) + \omega(LX, Y) = 0. \tag{9}$$

Of course, one can also start with ω and define $g(X, Y) = \omega(L^{-1}X, Y)$, then show that imposing compatibility of ω and L via (9) leads to the desired symmetry of g. Finally, given the knowledge of both g and ω, the bundle isomorphism L defined by (8) is uniquely determined, which satisfies (7), (9) and $L^2 = \pm\text{id}$. Whether L takes the form of J or K depends on whether (5) as opposed to (6) is to be satisfied.

In any case, the three objects g, ω and L with $L^2 = \pm id$ form a *compatible triple* such that given any two, the third one is rigidly "interlocked".

2.3 Klein Group of Transformations on ∇

We now show a key relationship between the three transformations of a connection ∇: its g-conjugate ∇^*, its ω-conjugate $\nabla^\dagger$, and its L-gauge transform ∇^L.

Theorem 1 ([8], Theorem 2.13) *Let (g, ω, L) be a compatible triple, and ∇^*, $\nabla^\dagger$, and ∇^L denote, respectively, g-conjugation, ω-conjugation, and L-gauge transformation of an arbitrary connection ∇. Then, $(id, *, \dagger, L)$ realizes a 4-element Klein group action on the space of affine connections:*

$$\begin{aligned}
(\nabla^*)^* &= (\nabla^\dagger)^\dagger = (\nabla^L)^L = \nabla;\\
\nabla^* &= (\nabla^\dagger)^L = (\nabla^L)^\dagger;\\
\nabla^\dagger &= (\nabla^*)^L = (\nabla^L)^*;\\
\nabla^L &= (\nabla^*)^\dagger = (\nabla^\dagger)^*.
\end{aligned}$$

Theorem 1 and Proposition 3, part (2) immediately lead to

Corollary 3 *Given a compatible triple (g, ω, L), $\nabla\omega = 0$ if and only if*

$$\nabla^* = \nabla^L.$$

Explicitly written,

$$\nabla^*_Z X = \nabla_Z X + L^{-1}((\nabla_Z L)X) = \nabla_Z X + L((\nabla_Z L^{-1})X). \tag{10}$$

Remark 2 Note that, in both Theorem 1 and Corollary 3, there is no requirement of ∇ to be torsion-free nor is there any assumption about its Codazzi coupling with L or with g. In particular, Corollary 3 says that, when viewing $\omega(X, Y) = g(LX, Y)$, $\nabla\omega = 0$ if and only if the torsions introduced by $*$ and by L are cancelled.

There have been confusing statements about (10), even for the special case of $L = J$, the almost complex structure. In Ref. [11, Proposition 2.5(2)], (10) was shown after assuming (g, ∇) to be a statistical structure. On the other hand, [24, Lemma 4.2] claimed the converse, also under the assumption of $(\mathfrak{M}, g, \nabla)$ being statistical. As Corollary 3 shows, the Codazzi coupling of ∇ and g is not relevant for (10) to hold; (10) is entirely a consequence of $\nabla\omega = 0$. Corollary 3 is a special case of a more general theorem ([31], Theorem 21).

2.4 Compatible Quadruple (g, ω, L, ∇)

We now consider simultaneous Codazzi couplings by the same ∇ with a compatible triple (g, ω, L). We first have the following result.

Theorem 2 *Let ∇ be a torsion-free connection on $\mathfrak{M}$, and $L = J, K$. Consider the following three statements regarding any compatible triple (g, ω, L)*

(i) (∇, g) *is Codazzi-coupled;*
(ii) (∇, L) *is Codazzi-coupled;*
(iii) $\nabla\omega = 0$.

Then

1. *Given (iii), then (i) and (ii) imply each other;*
2. *Assume ∇ is torsion-free, then (i) and (ii) imply (iii).*

Proof First, assuming (iii), we show that (i) and (ii) imply each other. This is because by Theorem 1, (iii) amounts to $\nabla = \nabla^\dagger$. Therefore, $\nabla^* = \nabla^L$. Hence: $T^{\nabla^*} = T^\nabla$ iff $T^{\nabla^L} = T^\nabla$. By Proposition 3 part (1), $T^{\nabla^*} = T^\nabla$ is equivalent to (g, ∇) being Codazzi coupled. By Lemma 1, $T^{\nabla^L} = T^\nabla$ is equivalent to (L, ∇) being Codazzi coupled. Hence, we proved that (i) and (ii) imply each other.

Next, assuming (i) and (ii), (iii) holds under the condition that ∇ is torsion-free. The proof is much involved, see the proof of Theorem 3.4 of [8].

In [8], we propose the notion of "compatible quadruple" to describe the compatibility between the four objects g, ω, L, and ∇ on a manifold $\mathfrak{M}$.

Definition 3 ([8], Definition 3.9) A *compatible quadruple* on $\mathfrak{M}$ is a quadruple (g, ω, L, ∇), where g and ω are symmetric and skew-symmetric non-degenerate (0,2)-tensors respectively, L is either an almost complex or almost para-complex structure, and ∇ is a torsion-free connection, that satisfy the following relations:

(i) $\omega(X, Y) = g(LX, Y)$;
(ii) $g(LX, Y) + g(X, LY) = 0$;
(iii) $\omega(LX, Y) = \omega(LY, X)$;
(iv) $(\nabla_X L)Y = (\nabla_Y L)X$;
(v) $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$;
(vi) $(\nabla_X \omega)(Y, Z) = 0$.

for any vector fields X, Y, Z on $\mathfrak{M}$.

As a consequence of Theorem 2, we have the following proposition regarding compatible quadruple.

Proposition 5 *Given a torsion-free connection* ∇, (g, ω, L, ∇) *forms a compatible quadruple if any of the following conditions holds:*

1. (g, L, ∇) *satisfy (ii), (iv) and (v);*
2. (ω, L, ∇) *satisfy (iii), (iv) and (vi);*
3. (g, ω, ∇) *satisfy (v) and (vi), in which case* L *is determined by (i).*

In other words, compatibility of ∇ with any two objects of the compatible triple makes a compatible quadruple, i.e., satisfying the three conditions as specified by either (1), (2), or (3) will lead to the satisfaction of all conditions (i)–(vi) of Definition 3.

2.5 Role of Connection ∇

A manifold $\mathfrak{M}$ admitting a compatible quadruple (g, ω, L, ∇), when ∇ is furthermore torsion-free, is in fact a (para-)Kähler manifold. This is because:

1. Codazzi coupling of L with a torsion-free ∇ ensures that L is integrable;
2. $\nabla\omega = 0$ with ∇ torsion-free ensures that $d\omega = 0$ (see Lemma 3.1 of [8]).

So the existence of a torsion-free connection ∇ on $\mathfrak{M}$ that is Codazzi couple to the compatible triple (g, ω, L) on $\mathfrak{M}$ gives rise to a (para-)Kähler structure on $\mathfrak{M}$.

Let us recall definitions of various types of structures on a manifold. A manifold $(\mathfrak{M}, g, L)$ where g is a Riemannian metric is said to be *almost (para-)Hermitian* if g and L are compatible; when furthermore L is integrable, then $(\mathfrak{M}, g, L)$ is called a *(para-)Hermitian* manifold. On the other hand, a manifold $(\mathfrak{M}, \omega)$ with a nondegenerate 2-form ω is said to be *symplectic* if we require ω to be closed, $d\omega = 0$. Amending $(\mathfrak{M}, \omega)$ with a (non-necessarily integrable) L turns $(\mathfrak{M}, \omega, L)$ into an *almost (para-)Kähler manifold* when L and ω are compatible. If furthermore we require both (i) an integrable L and (ii) a closed ω, then what we have on $\mathfrak{M}$ is a *(para-)Kähler structure*.

Note that in the definitions of (para-)Hermitian, symplectic, and (para-)Kähler structures, no affine connections are explicitly involved. In particular, (para-)Kähler manifold is defined by the integrability conditions of L and closedness of ω, which

440419_1_En_11_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

are related to topological properties of $\mathfrak{M}$. However, it is well-known in (para-)Kähler geometry that $(\mathfrak{M}, g, L)$ is (para-)Kähler if and only if L is parallel under the Levi-Civita connection of g, i.e., if there exists a torsion-free connection ∇ such that

$$\nabla g = 0, \quad \nabla L = 0.$$

So the existence of a "nice enough" connection on $\mathfrak{M}$ will enable a (para-)Kähler structure on it.

One the other hand, a *symplectic connection* ∇ is a connection that is both torsion-free and parallel to ω: $\nabla\omega = 0$. A symplectic manifold $(\mathfrak{M}, \omega)$, where $d\omega = 0$, equipped with a symplectic connection is known as a *Fedosov manifold* [14]. Since the parallelism of L with respect to any torsion-free ∇ implies that L is integrable, a symplectic manifold $(\mathfrak{M}, \omega)$ can be enhanced to a (para-)Kähler manifold if any symplectic connection on $\mathfrak{M}$ also renders L parallel:

$$\nabla\omega = 0, \quad \nabla L = 0.$$

Again, it is the existence of a "nice enough" connection that enhances the symplectic manifold to a (para-)Kähler manifold.

The contribution of our work is to extend the involvement of a connection ∇ from "parallelism" to "Codazzi coupling"; this is how statistical manifolds extend Riemannian manifolds. To this end, Theorem 2 says that for an arbitrary statistical manifold $(\mathfrak{M}, g, \nabla)$, if there exists an almost (para-)complex structure L compatible with g such that (the necessarily torsion-free, by definition of a statistical manifold) ∇ and L are Codazzi-coupled, then what we have of $(\mathfrak{M}, g, L, \nabla)$ is a (para-)Kähler manifold.

Theorem 2 also says that, for any Fedosov manifold $(\mathfrak{M}, \omega, \nabla)$, if there exists an almost (para-)complex structure L compatible with ω such that (the necessarily torsion-free, by definition of symplectic connection of a Fedosov manifold) ∇ and L are Codazzi-coupled, then $(\mathfrak{M}, \omega, L, \nabla)$ is a (para-)Kähler manifold. In other words, Codazzi coupling of ∇ with L turns a statistical manifold *or* a Fedosov manifold into a (para-)Kähler manifold, which is then both statistical and symplectic.

Proposition 6 *Given compatible triple (g, ω, L) on a manifold $\mathfrak{M}$, then any two of the following three statements imply the third, meanwhile turning $\mathfrak{M}$ into a (para-) Kähler manifold:*

(i) $(\mathfrak{M}, g, \nabla)$ *is a statistical manifold;*
(ii) $(\mathfrak{M}, \omega, \nabla)$ *is a Fedosov manifold;*
(iii) (∇, L) *is Codazzi coupled.*

440419_1_En_11_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

2.6 Codazzi-(Para-)Kähler Structure

Insofar as a compatible quadruple (g, ω, L, ∇) gives rise to a special kind of (para-)Kähler manifold, where the torsion-free ∇ is integrated snugly into the compatible triple (g, ω, L), we can call such a manifold *Codazzi-(para-)Kähler manifold*.

More generally, since as seen from Proposition 4, integrability of L may result from the existence of an affine connection ∇ that is Codazzi coupled to L *under the condition that* ∇ *is torsion-free*, we can have the following definition.

Definition 4 ([8], *Definition 3.8*) An *almost Codazzi-(para-)Kähler manifold* $\mathfrak{M}$ is by definition an almost (para-)Hermitian manifold $(\mathfrak{M}, g, L)$ with an affine connection ∇ (not necessarily torsion-free) which is Codazzi-coupled to both g and L. If ∇ is torsion-free, then L is automatically integrable and ω is parallel, so in this case we will call $(\mathfrak{M}, g, L, \nabla)$ a *Codazzi-(para-)Kähler manifold* instead.

So an almost Codazzi-(para-)Kähler manifold is an almost (para-)Hermitian manifold with a specified nice affine connection. Such structure exists on all almost (para-)Hermitian manifolds $(\mathfrak{M}, g, L)$. In particular, one can take ∇ to be any (para-)Hermitian connection [12, 18], which satisfies

$$\nabla g = 0 \text{ and } \nabla L = 0.$$

In the like manner, any (para-)Kähler manifold is trivially Codazzi-(para-)Kähler, because one can always take its Levi-Civita connection to be the desired ∇, turning the compatible triple into a compatible quadruple.

In a Codazzi-(para-)Kähler manifold, because of $\nabla\omega = 0$ which leads to $\nabla = \nabla^\dagger$ (Theorem 1), so $\nabla^* = \nabla^L$. Therefore, any Codazzi-(para-)Kähler manifold admits a pair (∇, ∇^C) of torsion-free connections, where ∇^C is called the *Codazzi dual* of ∇:

$$\nabla^C = \nabla^* = \nabla^L.$$

Proposition 7 *For any Codazzi-(para-)Kähler manifold, its Codazzi dual connection* ∇^C *satisfies:*

(i) $(\nabla^C_X L)Y = (\nabla^C_Y L)X$;
(ii) $(\nabla^C_X g)(Y, Z) = (\nabla^C_Y g)(X, Z)$;
(iii) $(\nabla^C_X \omega)(Y, Z) = 0$.

Introducing a family of α-connections for $\alpha \in \mathbb{R}$

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1+\alpha}{2}\nabla^C, \qquad \text{with} \quad (\nabla^{(\alpha)})^C = \nabla^{(-\alpha)}.$$

Then, we can easily show

(i) $(\nabla_X^{(\alpha)} L)Y = (\nabla_Y^{(\alpha)} L)X$;
(ii) $(\nabla_X^{(\alpha)} g)(Y, Z) = (\nabla_Y^{(\alpha)} g)(X, Z)$;
(iii) $(\nabla_X^{(\alpha)} \omega)(Y, Z) = 0$.

Remark 3 When $\alpha = 0$, this is the familiar case of Levi-Civita connection (which is also the Chern connection) on the (para-)Kähler manifold. We can see here that the entire family of α-connections are compatible with the same Codazzi-(para-)Kähler structure.

Let us now investigate how to enhance a statistical structure to Codazzi-(para-)Kähler structure. To this end, we have:

Theorem 3 *Let ∇ be a torsion-free connection on $\mathfrak{M}$, and $(\nabla^*, \nabla^\dagger, \nabla^L)$ are the transformations of ∇ induced by the compatible triple (g, ω, L). Then (g, ω, L, ∇) forms a compatible quadruple if any two of the following three statements are true:*

(i) ∇^* is torsion-free;
(ii) $\nabla^\dagger$ is torsion-free;
(iii) ∇^L is torsion-free.

The proof is rather straight-forward, invoking Proposition 3 and Lemma 1 which link Codazzi coupling condition to torsion preservation in conjugate and gauge transformations.

In this case, i.e., when any two of the above three statements are true, $\mathfrak{M}$ is Codazzi-(para-)Kähler. Hence, Theorem 3 can be viewed as a characterization theorem for Codazzi-(para-)Kähler structure, in the same way that condition (i) above alone characterizes statistical structure (a la Lauritzen [21]). This provides the affirmative answer to the key question posed by our paper: A statistical structure $(\mathfrak{M}, g, \nabla)$ can be "enhanced" to a Codazzi-(para-)Kähler structure $(\mathfrak{M}, g, \omega, L, \nabla)$ by

1. supplying it with an L that is compatible with g and that is Codazzi coupled with ∇;
2. supplying it with an L that is compatible with g and such that ∇^L is torsion-free; or
3. supplying it with an ω such that $\nabla\omega = 0$.

To summarize, in relation to more familiar types of manifolds, a Codazzi-(para-) Kähler manifold is a (para-)Kähler manifold which is at the same time statistical; it is also a Fedosov (hence symplectic) manifold which is at the same time statistical.

3 Divergence Functions and (Para-)Kähler Structures

Roughly speaking, a divergence function provides a measure of "directed distance" between two probability distributions in a family parameterized by a manifold $\mathfrak{M}$. Starting from a (local) divergence function on $\mathfrak{M}$, there are standard techniques to generate a statistical structure on the diagonal $\mathfrak{M}_\Delta := \{(x, y) \in \mathfrak{M} \times \mathfrak{M} : x = y\} \subset \mathfrak{M} \times \mathfrak{M}$ as well as a symplectic structure on $\mathfrak{M} \times \mathfrak{M}$. We will first review these techniques and then show that para-Kähler structures on $\mathfrak{M} \times \mathfrak{M}$ arise naturally in this setting. Kähler structures will also be discussed. In the end, we study the case where $\mathfrak{M}$ is Kähler and the local divergence function is taken to be Calabi's diastatic function. Its very rich geometric structures can be built in this scenario.

3.1 Classical Divergence Functions and Statistical Structures

Definition 5 (*Classical divergence function*) Let $\mathfrak{M}$ be a smooth manifold of dimension n. A *classical divergence function* is a non-negative smooth function $\mathcal{D} : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}_{\geq 0}$ satisfying the following conditions:

(i) $\mathcal{D}(x, y) \geq 0$ for any $(x, y) \in \mathfrak{M} \times \mathfrak{M}$, with equality holds if and only if $x = y$;
(ii) The diagonal $\mathfrak{M}_\Delta \subset \mathfrak{M} \times \mathfrak{M}$ is a critical submanifold of $\mathfrak{M}$ with respect to $\mathcal{D}$, in other words, $\mathcal{D}_i(x, x) = \mathcal{D}_{,j}(x, x) = 0$ for any $1 \leq i, j \leq n$;
(iii) $-\mathcal{D}_{i,j}(x, x)$ is positive definite at any $(x, x) \in \mathfrak{M}_\Delta$.

Here $\mathcal{D}_i(x, y) = \partial_{x^i}\mathcal{D}(x, y)$, $\mathcal{D}_{,j}(x, y) = \partial_{y^j}\mathcal{D}(x, y)$, and $\mathcal{D}_{i,j}(x, y) = \partial_{x^i}\partial_{y^j}\mathcal{D}(x, y)$ and so on, where $\{x^i\}_{i=1}^n$ and $\{y^j\}_{j=1}^n$ are local coordinates of $\mathfrak{M}$ near x and y respectively. When $x = y$, we further require that $\{x^i\}_{i=1}^n$ and $\{y^j\}_{j=1}^n$ give the same coordinates on $\mathfrak{M}$. Under such assumption, one can easily check that properties (i), (ii) and (iii) are independent of the choice of local coordinates. Note that $\mathcal{D}$ does not have to satisfy $\mathcal{D}(x, y) = \mathcal{D}(y, x)$.

A standard example of classical divergence function is the Bregman divergence [3]. Given any smooth and strictly convex function $\Phi : \Omega \to \mathbb{R}$ on a closed convex set Ω, the Bregman divergence $\mathcal{B}_\Phi : \Omega \times \Omega \to \mathbb{R}$ is defined by

$$\mathcal{B}_\Phi(x, y) = \Phi(x) - \Phi(y) - \langle x - y, \nabla\Phi(y)\rangle \tag{11}$$

where $\nabla\Phi$ is the usual gradient of Φ and $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$. Generalizing Bregman's divergence, we have the following Φ-divergence for all $\alpha \in \mathbb{R}$ [35]:

$$\mathcal{D}_\Phi^{(\alpha)}(x, y) = \frac{4}{1-\alpha^2}\left(\frac{1-\alpha}{2}\Phi(x) + \frac{1+\alpha}{2}\Phi(y) - \Phi\left(\frac{1-\alpha}{2}x + \frac{1+\alpha}{2}y\right)\right). \tag{12}$$

It is known [7] that a statistical structure on $\mathfrak{M}_\Delta$ can be induced from a classical divergence function $\mathcal{D}$. Consider the Taylor expansion of $\mathcal{D}$ along $\mathfrak{M}_\Delta$, we obtain:

(i) (2nd order): a Riemannian metric g

$$g_{ij}(x) = -\mathcal{D}_{i,j}(x, x) = \mathcal{D}_{ij}(x, x) = -\mathcal{D}_{j,i}(x, x).$$

440419_1_En_11_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

(ii) (3rd order): a pair of conjugate connections

$$\Gamma_{ij,k}(x) = -\mathcal{D}_{ij,k}(x,x), \quad \Gamma^*_{ij,k}(x) = -\mathcal{D}_{k,ij}(x,x).$$

One can verify that the definitions of g, ∇ and ∇^* are independent of the choice of coordinates and indeed ∇^* is the g-conjugate of ∇, i.e.,

$$\partial_k g_{ij} = \Gamma_{ki,j} + \Gamma^*_{kj,i}.$$

Moreover, ∇ is torsion-free and (∇, g) is Codazzi-coupled, so we obtain a statistical structure on $\mathfrak{M}_\Delta$.

As an example, from the Φ-divergence $\mathcal{D}^{(\alpha)}_\Phi(x,y)$, we get the α-Hessian structure on $\mathfrak{M}$ (see [35]) consisting of

$$g_{ij}(x) = \Phi_{ij}(x)$$

and

$$\Gamma^{(\alpha)}_{ij,k}(x) = \frac{1-\alpha}{2}\Phi_{ijk}(x), \quad \Gamma^{*(\alpha)}_{ij,k}(x) = \frac{1+\alpha}{2}\Phi_{ijk}(x).$$

3.2 Generalized Divergence Functions and Symplectic Structures

In this subsection, we will use a slightly different notion of divergence functions.

Definition 6 (*Generalized divergence function*) Let $\mathfrak{M}$ be a smooth manifold of dimension n. A *generalized divergence function* is a smooth function $\mathcal{D} : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}$ satisfying the following conditions:

(i) The diagonal $\mathfrak{M}_\Delta \subset \mathfrak{M} \times \mathfrak{M}$ is a critical submanifold of $\mathfrak{M}$ with respect to $\mathcal{D}$; in other words, $\mathcal{D}_i(x,x) = \mathcal{D}_{,j}(x,x) = 0$ for any $1 \le i, j \le n$;
(ii) $\mathcal{D}_{i,j}(x,y)$ is a nondegenerate matrix at any point $(x,y) \in \mathfrak{M} \times \mathfrak{M}$.

Once again, $\{x^i\}_{i=1}^n$ and $\{y^j\}_{j=1}^n$ are arbitrary local coordinates of $\mathfrak{M}$ near x and y respectively. It is obvious that this definition does not rely on the choice of local coordinates. Using the same ingredient as before, we can cook up a metric and a pair of conjugate torsion-free connections on $\mathfrak{M}_\Delta$. However this metric may be indefinite.

Barndorff-Nielsen and Jupp [2] associated a symplectic form on $\mathfrak{M} \times \mathfrak{M}$ with $\mathcal{D}$ (called "yoke" there), defined as (apart from a minus sign added here)

$$\omega_{\mathcal{D}}(x,y) = -\mathcal{D}_{i,j}(x,y)dx^i \wedge dy^j. \tag{13}$$

In particular, Bregman divergence $\mathcal{B}_\Phi$ (which fulfils the definition of a generalized divergence function) induces the symplectic form $\sum \Phi_{ij} dx^i \wedge dy^j$.

Such a construction essentially treated the divergence function as the Type II generating function of the symplectic structure on $\mathfrak{M} \times \mathfrak{M}$, see [22]. Let us consider the map $L_{\mathcal{D}} : \mathfrak{M} \times \mathfrak{M} \to T^*\mathfrak{M}$ given by

$$(x, y) \mapsto (x, \mathrm{d}(\mathcal{D}(\cdot, y))(x)) = \left(x, \sum_i \mathcal{D}_i(x, y)\mathrm{d}x^i \right).$$

Given y, we think of $\mathcal{D}(\cdot, y)$ as a smooth function of $x \in \mathfrak{M}$ and $\mathrm{d}(\mathcal{D}(\cdot, y))(x)$ is nothing but the value of its differential at point x.

Recall that $T^*\mathfrak{M}$ admits a canonical symplectic form ω_{can}. A local calculation shows that

$$\omega_{\mathcal{D}} = -L^*_{\mathcal{D}}\,\omega_{\text{can}}.$$

In addition, it is not hard to see that condition (ii) in Definition 6 of a $\mathcal{D}$ is equivalent to that $L_{\mathcal{D}}$ is a local diffeomorphism. Therefore $\omega_{\mathcal{D}}$ is indeed a symplectic form on $\mathfrak{M} \times \mathfrak{M}$. Similarly we can consider the map $R_{\mathcal{D}} : \mathfrak{M} \times \mathfrak{M} \to T^*\mathfrak{M}$ given by

$$(x, y) \mapsto (y, \mathrm{d}(\mathcal{D}(x, \cdot))(y)) = \left(y, \sum_j \mathcal{D}_{,j}(x, y)\mathrm{d}y^j \right).$$

In the same manner, we see that

$$\omega_{\mathcal{D}} = R^*_{\mathcal{D}}\,\omega_{\text{can}}.$$

Let $\mathfrak{M}_x = \{x\} \times \mathfrak{M}$ and $\mathfrak{M}_y = \mathfrak{M} \times \{y\}$. From the expression (13), we see immediately that $\mathfrak{M}_x$, $\mathfrak{M}_y$ and $\mathfrak{M}_\Delta$ are Lagrangian submanifolds of $(\mathfrak{M} \times \mathfrak{M}, \omega_{\mathcal{D}})$.

3.3 Para-Kähler Structure on $\mathfrak{M} \times \mathfrak{M}$

Let M be a smooth manifold and $\mathcal{D} : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}$ be a generalized divergence function per Definition 6. From (13), $\mathcal{D}$ induces a symplectic form $\omega_{\mathcal{D}}$ on $\mathfrak{M} \times \mathfrak{M}$. Actually, this symplectic form comes from a natural para-Kähler structure on $\mathfrak{M} \times \mathfrak{M}$ as we show below.

Let $(x, y) \in \mathfrak{M} \times \mathfrak{M}$ be an arbitrary point. Using the canonical identification

$$T_{(x,y)}(\mathfrak{M} \times \mathfrak{M}) = T_x\mathfrak{M} \oplus T_y\mathfrak{M},$$

we can produce an almost para-complex structure K on $\mathfrak{M} \times \mathfrak{M}$ by assigning

$$K_{(x,y)} = \mathrm{id} \text{ on } T_x\mathfrak{M} \oplus 0 \quad \text{and} \quad K_{(x,y)} = -\mathrm{id} \text{ on } 0 \oplus T_y\mathfrak{M}.$$

440419_1_En_11_Chapter ✓ TYPESET ☐ DISK ☐ LE ✓ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

It is clear from this definition that K is integrable. Moreover, it can be checked that K and $\omega_\mathcal{D}$ are compatible in the sense that

$$\omega_\mathcal{D}(KX, KY) = -\omega_\mathcal{D}(X, Y). \tag{14}$$

Therefore the associated metric $g(X, Y) = \omega(KX, Y)$ is also compatible with K and we get a para-Kähler structure on $\mathfrak{M} \times \mathfrak{M}$.

Now let E_1 and E_{-1} be the eigen-distributions of K of eigenvalues 1 and -1 respectively. We see instantly that they are Lagrangian foliations with leaves $\mathfrak{M}_x$'s and $\mathfrak{M}_y$'s respectively.

Note that (14) does not impose any restriction on the form of the generalized divergence function $\mathcal{D}$. So we have the following structure theorem of manifolds admitting a generalized divergence function.

Theorem 4 *Let $\mathfrak{M}$ be a smooth manifold admitting a generalized divergence function $\mathcal{D}$. Then $\mathfrak{M}$ must be orientable, non-compact and parallelizable. Moreover, M supports an affine structure, i.e., there exists a torsion-free flat connection on $\mathfrak{M}$.*

Proof Assuming the existence of $\mathcal{D}$, we can produce a symplectic form $\omega_\mathcal{D}$ on $\mathfrak{M} \times \mathfrak{M}$ as in the last subsection. Therefore $\mathfrak{M} \times \mathfrak{M}$ is orientable and so is $\mathfrak{M}$. Moreover, $\omega_\mathcal{D}$ is an exact symplectic form since it is pull-back of an exact symplectic form (the canonical symplectic form on a cotangent bundle), therefore M cannot be compact.

As $E_{\pm 1}$ are Lagrangian foliations, it follows from Weinstein's result [34] that $\mathfrak{M}$, diffeomorphic to a leaf of a Lagrangian foliation, is affine. Indeed, such torsion-free flat connections can be construct explicitly on $\mathfrak{M}$. Let ∇^{LC} be the Levi-Civita connection associated to the para-Kähler structure $(\mathfrak{M} \times \mathfrak{M}, K, g)$. A straightforward calculation (see [16] and [32, Proposition 3.2]) shows that the connections induced by ∇^{LC} on leaves of $E_{\pm 1}$ are flat. Therefore, by identifying $\mathfrak{M}$ with $\mathfrak{M}_x(\mathfrak{M}_y)$ for varying $x(y)$, we actually obtain two families of affine structure on $\mathfrak{M}$ parameterized by $\mathfrak{M}$ itself.

Finally, as $T_x\mathfrak{M}$ and $T_y\mathfrak{M}$ are Lagrangian subspaces of $(T_{(x,y)}(\mathfrak{M} \times \mathfrak{M}), \omega_\mathcal{D})$, we obtain an isomorphism

$$T_x\mathfrak{M} \cong (T_y\mathfrak{M})^*$$

using ω_D. If we fix $y = y_0$, then we get a smooth identification

$$T_x\mathfrak{M} \cong (T_{y_0}\mathfrak{M})^* \cong T_{x'}\mathfrak{M}$$

for any $x, x' \in \mathfrak{M}$, which parallelize $T\mathfrak{M}$.

Remark 4 The signature (n, n) of the pseudo-Riemannian metric g on $\mathfrak{M} \times \mathfrak{M}$ can be written down explicitly as

$$g = -\mathcal{D}_{i,j} \mathrm{d}x^i \otimes \mathrm{d}y^j.$$

Therefore the induced metric on $\mathfrak{M}_\Delta$ agrees with the metric constructed by 2nd order expansion of $\mathcal{D}$ in Sect. 3.1. However in general, the pair of conjugate connections ∇ and ∇^* on $\mathfrak{M}_\Delta$ constructed from 3rd order expansion are distinct, therefore they do not coincide with ∇^{LC} associated to g.

In fact, we can give a full characterization of manifold with generalized divergence functions.

Theorem 5 *An n-dimensional manifold $\mathfrak{M}$ admits a generalized divergence function $\mathcal{D}$ if and only if M can be immersed into $\mathbb{R}^n$.*

Proof Fix a point $y_0 \in \mathfrak{M}$ and linear independent tangent vectors $v_1, \ldots, v_n \in T_{y_0}\mathfrak{M}$. If $\mathfrak{M}$ admits a generalized divergence function $\mathcal{D}$, we can consider the map $f : \mathfrak{M} \to \mathbb{R}^n$ given by

$$f(x) = (v_1\mathcal{D}(x, y_0), \ldots, v_n\mathcal{D}(x, y_0)).$$

Then by the nondegeneracy condition of $\mathcal{D}$, we know that f has invertible Jacobian, hence it is an immersion.

On the other hand, $\mathcal{D}_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $\mathcal{D}_0(x, y) = x \cdot y$ is a generalized divergence function on $\mathbb{R}^n$. If $\mathfrak{M}$ can be immersed into $\mathbb{R}^n$, then we can pull-back $\mathcal{D}_0$ to get a generalized divergence function on $\mathfrak{M}$.

Remark 5 All the results of Theorem 4 follow trivially from Theorem 5. However, we state it independently because the constructions in the proof of Theorem 4 are canonical. It should also be noted that the condition that $\mathfrak{M}$ can be immersed into $\mathbb{R}^n$ is a much weaker than that $\mathfrak{M}$ can be imbedded as an open subset of $\mathbb{R}^n$. For example, if we let $\mathfrak{M} = (S^2 \times S^1) \setminus \{\mathrm{pt}\}$, then $\mathfrak{M}$ can be immersed into $\mathbb{R}^3$ but not imbedded into it.

Para-complex manifolds have very rich geometric structures. For instance, one can recognize various Dirac structures [5] on them. Let $\mathfrak{N}$ be a smooth manifold of dimension n. Following Courant, we define a Dirac structure on $\mathfrak{N}$ as a rank n subbundle of $T\mathfrak{N} \oplus T^*\mathfrak{N}$ which is closed under the Courant bracket $[\cdot, \cdot]_C$:

$$[X \oplus \xi, Y \oplus \eta]_C = [X, Y] \oplus (\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\mathrm{d}(\iota_X\eta - \iota_Y\xi))$$

for any smooth vector fields X, Y and 1-forms ξ, η, where ι is the interior product and $\mathcal{L}$ is the Lie derivative. If $(\mathfrak{N}, K)$ is a para-complex manifold, then we have the decomposition

$$T\mathfrak{N} = E_1 \oplus E_{-1},$$

where $E_{\pm 1}$ are eigen-distributions of eigenvalue 1 and -1 with respect to K. This splitting induces the decomposition for cotangent bundle:

440419_1_En_11_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:29/7/2018 Pages: 26 Layout: T1-Standard

$$T^*\mathfrak{N} = E_1^* \oplus E_{-1}^*.$$

It is not hard to check that $D_{\pm 1} = E_{\pm 1} \oplus E_{\mp 1}^*$ define two transversal Dirac structures on $\mathfrak{N}$. In particular, we obtain such structures on $\mathfrak{M} \times \mathfrak{M}$ if $\mathfrak{M}$ admits a generalized divergence function. It is of great interest to understand the statistical interpretation of Courant bracket, as well as Dirac structures. We also refer to [33] for a general discussion of para-complex manifolds and Dirac structures.

3.4 Local Divergence Functions and Kähler Structures

A natural question to ask is whether one can construct a Kähler structure on $\mathfrak{M} \times \mathfrak{M}$ from a divergence function on $\mathfrak{M}$. The first problem one has to solve is to construct a complex structure J on $\mathfrak{M} \times \mathfrak{M}$. Unlike the para-complex case, there seems to be no canonical choice of J. So instead, we only consider a local version of this problem, i.e., constructing a Kähler structure on a neighborhood of $\mathfrak{M}_\Delta$ inside $\mathfrak{M} \times \mathfrak{M}$.

Definition 7 (*Local divergence function*) Let $\mathfrak{M}$ be a smooth manifold of dimension n. A *local divergence function* is a nonnegative smooth function $\mathcal{D}$ defined on an open neighborhood U of $\mathfrak{M}_\Delta$ inside $\mathfrak{M} \times \mathfrak{M}$ such that

(i) $\mathcal{D}(x, y) \geq 0$ for any $(x, y) \in U$, with equality holds if and only if $x = y$;
(ii) The diagonal $\mathfrak{M}_\Delta$ is a critical submanifold of $\mathfrak{M}$ with respect to $\mathcal{D}$, in other words, $\mathcal{D}_i(x, x) = \mathcal{D}_{,j}(x, x) = 0$ for any $1 \leq i, j \leq n$;
(iii) $-\mathcal{D}_{i,j}(x, x)$ is positive definite at any $(x, x) \in \mathfrak{M}_\Delta$.

It is obvious from this definition that classical divergence functions are local divergence functions. On the other hand, by a partition of unity argument, one can always extend a local divergence function to a classical divergence function. And moreover, local divergence is indeed a local version of divergence function we defined in Sect. 3.2.

To define a complex structure on a neighborhood of $\mathfrak{M}_\Delta$, let us assume that $\mathfrak{M}$ is an affine manifold, i.e., there exists a coordinate cover of $\mathfrak{M}$ such that coordinate transformations are affine transformations. Let $\{U_\alpha\}_\alpha$ be the set of affine coordinate charts on $\mathfrak{M}$. Then

$$U = \bigcup_\alpha U_\alpha \times U_\alpha \subset \mathfrak{M} \times \mathfrak{M}$$

is an open neighborhood of $\mathfrak{M}_\Delta$. We can define a complex structure J on U as follows. For any point $(x, y) \in U_\alpha \times U_\alpha \subset U$, we define J by assigning

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i} \quad \text{and} \quad J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}$$

for $1 \leq i \leq n$, here $\{x^i\}_{i=1}^n, \{y^i\}_{i=1}^n$ are two copies of the same coordinates on U_α. As a consequence of $\mathfrak{M}$ being affine, J does not depend on the choice of U_α. Furthermore, J is integrable since we may use $z^j = x^j + iy^j$ as holomorphic coordinates. However in general, J cannot be extended to a complex structure on $\mathfrak{M} \times \mathfrak{M}$.

Analogous to the para-Kähler case (14), we would like to have the compatibility condition between $\omega_\mathcal{D}$ and J

$$\omega_\mathcal{D}(JX, JY) = \omega_\mathcal{D}(X, Y).$$

As $\omega_\mathcal{D}$ is induced by the generalized divergence function $\mathcal{D}$ via (13), the above condition does impose a restriction on the generalized divergence function $\mathcal{D}$

$$\mathcal{D}_{i,j} = \mathcal{D}_{j,i}$$

or explicitly

$$\frac{\partial^2 \mathcal{D}}{\partial x^i \partial y^j} = \frac{\partial^2 \mathcal{D}}{\partial y^i \partial x^j}.$$

We call such divergence functions "proper". This condition was first derived in Zhang and Li [37]. As an example, the Φ-divergence given in (12) satisfies this condition of properness.

Now let us take the local proper divergence function $\mathcal{D}$ into account. Using $\mathcal{D}$ as a Kähler potential, we obtain

$$i\partial\bar{\partial}\mathcal{D} = \frac{i}{4}(\mathcal{D}_{jk} + \mathcal{D}_{,jk} + i\mathcal{D}_{j,k} - i\mathcal{D}_{k,j})\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k = \frac{i}{4}(\mathcal{D}_{jk} + \mathcal{D}_{,jk})\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k.$$

When restricting to $\mathfrak{M}_\Delta$, we see that

$$\mathcal{D}_{jk}(x, x) + \mathcal{D}_{,jk}(x, x) = -2\mathcal{D}_{j,k}(x, x)$$

form a positive definite matrix. Therefore in a sufficiently small open neighborhood U, the (1,1)-form $i\partial\bar{\partial}\mathcal{D}$ is Kähler and we obtain a Kähler structure on U whose restriction on $\mathfrak{M}_\Delta$ agrees with the original metric on $\mathfrak{M}$ up to a scalar.

3.5 An Example: The Case of Analytic Kähler Manifold

When $\mathfrak{M}$ itself is an analytic Kähler manifold, we have a canonical choice of local divergence function: the diastatic function defined by Calabi [4].

Let $(\mathfrak{M}, I_0, \Omega_0)$ be an analytic Kähler manifold, that is, $\mathfrak{M}$ is a Kähler manifold with complex structure I_0 such that the Kähler metric Ω_0 is real analytic with respect to the natural analytic structure on $\mathfrak{M}$. Let $\overline{\mathfrak{M}}$ be the conjugate manifold of $\mathfrak{M}$. By this, we mean a complex manifold related to $\mathfrak{M}$ by a diffeomorphism mapping $p \in \mathfrak{M}$ onto a point $\bar{p} \in \overline{\mathfrak{M}}$, such that for each local holomorphic coordinate $\{z^1, \dots, z^n\}$ in a neighborhood V of p, there exists a local holomorphic coordinate $\{w^1, \dots, w^n\}$ in the image $\bar{V}$ of V, satisfying

$$w^j(\bar{q}) = \overline{z^j(q)}, \text{ for } j = 1, \dots, k.$$

Abstractly, $\overline{\mathfrak{M}}$ is the complex manifold $(\mathfrak{M}, -I_0)$ with local holomorphic coordinates of specified as above.

Let Ψ be a Kähler potential of Ω_0, that is, Ψ is a locally defined real-valued function such that $i\partial\overline{\partial}\Psi = \Omega_0$. In local coordinates on V, we have

$$\Omega_0 = i\frac{\partial^2\Psi(z,\overline{z})}{\partial z^j \partial \overline{z}^k} \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^k.$$

As by our assumption Ω_0 is real analytic, so is Ψ, therefore in a small enough neighborhood Ψ can be written as a convergent power series of z and $\overline{z}$. Think of $\overline{z}$ as coordinates on $\overline{\mathfrak{M}}$, then using this power series expansion, Ψ is a local holomorphic function on $\mathfrak{M} \times \overline{\mathfrak{M}} \cong \mathfrak{M} \times \mathfrak{M}$ defined in a neighborhood U of diagonal $\mathfrak{M}_\Delta$.

Calabi defined the diastatic function $\mathcal{D}_d : U \to \mathbb{R}$ by

$$\mathcal{D}_d(p, \bar{q}) = \Psi(p, \bar{p}) + \Psi(q, \bar{q}) - \Psi(p, \bar{q}) - \Psi(q, \bar{p}). \tag{15}$$

Using our language, Calabi essentially proved the following theorem:

Theorem 6 ([4, Proposition 1–5]) *The diastatic function $\mathcal{D}_d$ defined by (15) does not depend on the choice of local holomorphic coordinate.*

In other words, $\mathcal{D}_d$ is a local divergence function. Now we use $\mathcal{D}_d$ to perform the constructions in previous sections.

In local coordinates, write $z^j = x^j + iy^j$ and $w^j = u^j - iv^j$. Due to the complex conjugation we need to identify $\overline{\mathfrak{M}}$ with $(\mathfrak{M}, -I_0)$, we have that $\{x^j, y^k\}_{j,k=1}^n$ and $\{u^j, v^k\}_{j,k=1}^n$ form two copies of identical coordinates. As

$$\mathcal{D}_d(x, y, u, v) = \Psi(z, \overline{z}) + \Psi(\overline{w}, w) - \Psi(z, w) - \Psi(\overline{w}, \overline{z}),$$

we can compute directly that

$$\begin{aligned}
\frac{\partial^2 \mathcal{D}_d}{\partial x^j \partial u^k} &= -\frac{\partial^2 \Psi}{\partial z^j \partial \overline{z}^k}(z, w) - \frac{\partial^2 \Psi}{\partial z^k \partial \overline{z}^j}(\overline{w}, \overline{z}) = \frac{\partial^2 \mathcal{D}_d}{\partial y^j \partial v^k}, \\
\frac{\partial^2 \mathcal{D}_d}{\partial x^j \partial v^k} &= i\frac{\partial^2 \Psi}{\partial z^j \partial \overline{z}^k}(z, w) - i\frac{\partial^2 \Psi}{\partial z^k \partial \overline{z}^j}(\overline{w}, \overline{z}) = -\frac{\partial^2 \mathcal{D}_d}{\partial y^j \partial u^k}.
\end{aligned}$$

Therefore the holomorphic symplectic form, as induced via (13), is given by

$$\begin{aligned}\Omega &= (\Psi_{j\bar{k}}(z,w) + \Psi_{k\bar{j}}(\overline{w},\overline{z}))(\mathrm{d}x^j \wedge \mathrm{d}u^k + \mathrm{d}y^j \wedge \mathrm{d}v^k) - i(\Psi_{j\bar{k}}(z,w) \\ &\quad - \Psi_{k\bar{j}}(\overline{w},\overline{z}))(\mathrm{d}x^j \wedge \mathrm{d}v^k - \mathrm{d}y^j \wedge \mathrm{d}u^k) \\ &= \Psi_{j\bar{k}}(z,w)\mathrm{d}z^j \wedge \mathrm{d}w^k + \Psi_{k\bar{j}}(\overline{w},\overline{z})\mathrm{d}\overline{z}^j \wedge \mathrm{d}\overline{w}^k \\ &= \Omega_{\mathbb{C}} + \overline{\Omega_{\mathbb{C}}},\end{aligned}$$

where $\Omega_{\mathbb{C}} = \Psi_{j\bar{k}}(z,w)\mathrm{d}z^j \wedge \mathrm{d}w^k$ is a well-defined complex-valued 2-form.

There are two natural complex structures on $\mathfrak{M} \times \mathfrak{M} \cong \mathfrak{M} \times \overline{\mathfrak{M}}$, i.e., $J^+ := (I_0, I_0)$ and $J^- := (I_0, -I_0)$, whose holomorphic coordinates are given by $\{z^j, \overline{w}^k\}_{j,k=1}^n$ and $\{z^j, w^k\}_{j,k=1}^n$ respectively.

It is clear from the above expression of Ω that Ω is a (1,1)-form with respect to J^+, therefore (U, J^+, Ω) is a pseudo-Kähler manifold such that $\mathfrak{M}_\Delta$ is a Lagrangian submanifold. On the other hand, with respect to J^-, we see that $\Omega_{\mathbb{C}}$ is a holomorphic symplectic form whose restriction on $\mathfrak{M}_\Delta$ is the Kähler form Ω_0 on $\mathfrak{M}$ up to a purely imaginary scalar. It was proved years ago independently by Kaledin [19] and Feix [9], using different methods, that U actually admits a hyperkähler metric.

Notice that J^+ commutes with J^- and $-J^+J^- = K$ is the para-complex structure we specified in Sect. 3.3. A manifold with such structures was used by [13] and many other places in string theory as "modified Calabi–Yau manifolds", see [26] for more details.

If we further assume that $\mathfrak{M}$ is also affine in the sense that the holomorphic coordinates on $\mathfrak{M}$ change by affine transformations, then we can use the recipe in Sect. 3.4 to construct a complex structure J on U with Kähler metric $i\partial\overline{\partial}\mathcal{D}_d$. To be specific, $\{x^j + iu^j, y^k + iv^k\}_{j,k=1}^n$ gives local holomorphic coordinates on U with respect to J. It is straightforward to see that J^+ commutes with J while J^- anticommutes with J, which leads to a modified Calabi–Yau structure and a hypercomplex structure on U, respectively.

4 Discussions

Codazzi coupling is the cornerstone of affine differential geometry (e.g., [25, 30]), and in particular so for information geometry. In information geometry, the Riemannian metric g and a pair of torsion-free g-conjugate affine connections ∇, ∇^* are naturally induced by the so-called divergence (or "contrast") function on a manifold $\mathfrak{M}$ (see [1]). While a statistical structure is naturally induced on $\mathfrak{M}$, the divergence function will additionally induce a symplectic structure ω on the product manifold $\mathfrak{M} \times \mathfrak{M}$, see [2, 37]. Reference [31] appears to be the first to extend the definition of conjugate connection with respect to g to that with respect to ω. And [8] proved that the g-conjugate, ω-conjugate, L-gauge transformations of ∇ form a Klein group. Based on these, it is shown that Codazzi coupling of torsion-free ∇ with any two of the compatible triple (g, ω, L) implies its coupling with the remaining third, turning (g, ω, L, ∇) into a compatible quadruple and hence the manifold $\mathfrak{M}$ into a (para-)Kähler one. Therefore, our results here provide precise conditions under which a statistical manifold could be “enhanced” to a Kähler and/or para-Kähler manifold, and clarify some confusions in the literature regarding the roles of Codazzi coupling of ∇ with g and with L in the interactions between statistical structure (as generalized Riemannian structure), symplectic structure, and (para-)complex structure.

Codazzi-(para-)Kähler manifolds are generalizations of special Kähler manifolds by removing the requirement of ∇ to be (dually) flat in the latter. Special Kähler manifolds are first mathematically formulated by Freed [10], and they have been extensively studied in physics literature since 1980s. For example, special Kähler structures are found on the base of algebraic integrable systems [6] and moduli space of complex Lagrangian submanifolds in a hyperkähler manifold [17]. From the above discussions, we can view special Kähler manifolds as “enhanced” from the class of dually-flat statistical manifold, namely, Hessian manifolds [29]. In information geometry, non-flat affine connections are abundant – the family of $\nabla^{(\alpha)}$ connections associated with a pair of dually-flat connections ∇, ∇^* are non-flat except $\alpha = \pm 1$ [36]. So our generalization of special Kähler geometry to Codazzi-Kähler geometry, which shifts attention from curvature to torsion, may be meaningful for the investigation of bidualistic geometric structures in statistical and information sciences [35].

Acknowledgements This collaborative research started while the first author (J.Z.) was on sabbatical visit at the Center for Mathematical Sciences and Applications at Harvard University in the Fall of 2014 under the auspices of Prof. S.-T. Yau. The writing of this paper is supported by DARPA/ARO Grant W911NF-16-1-0383 (PI: Jun Zhang).

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