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Null preference and the resolution of the topological social choice paradox

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HIGHLIGHTS

- Considered the role of null preference in social choice with non-Hausdorff topology of preference space.
- Constructed a contractible space from non-contractible one by adding the null preference point — the "null closure" procedure.
- Resolved the topological social choice paradox by proving an extension of Resolution Theorem.

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ABSTRACT

We investigate the role of contractibility in topological social choice theory. The Resolution Theorem states that there exists an aggregation map that is anonymous, unanimous, and continuous if and only if the space of individual preferences is contractible. Here, we turn a non-contractible space of social preferences (modeled as a CW complex) into a contractible space by adding the null preference which models full indifference of society, following the possibility results of Jones et al. (2003) which is based on a topology first considered but rejected by Le Breton and Uriarte (1990). We prove the corresponding extension of the Resolution Theorem by showing that the null preference as a social outcome precisely captures those voter profiles that represent a "tie" under a Chichilnisky map. Further, the space of these tie profiles is shown to have measure zero in the case that the space of individual preferences is a sphere in any dimension.

1. Introduction

Social choice theory aims to understand the nature of, and provide methods for, the aggregation of individual preferences to yield a social preference which is fair and satisfactory to individual voters on an axiomatic ground. An important example of this is in popular elections, where a large population must choose its favorite candidates. Where Arrow (1963) proved the nonexistence of rational choice for a finite set of alternatives, Chichilnisky and Heal (1983) (following Eckmann, 1954) set up a local differentiable framework on a n-dimensional cube of alternatives and proved that social welfare functions which are continuous, anonymous, and unanimous exist if and only if the space of preferences is contractible. For overviews, we refer the reader to Lauwers (2000) and Baigent (2011).

In this framework, the space of preferences is always assumed to be a CW complex. It is worth noting that the restriction to CW complexes is strong, but still leaves a very wide collection of possible spaces. Unfortunately, there are several noteworthy instances in previous literature where this requirement is ignored. It is incorrect to claim that any preference space has a Chichilnisky map if and only if it is contractible. It is crucial to recall that the theorem only applies to (parafinite) CW complexes.

Le Breton and Uriarte (1990) considered the idea of inserting a null preference for the special case of a preference space given by S^n, the n-sphere, which is a non-contractible finite CW complex. There, the new space $S^n \cup \{\mathcal{O}\}$ is allowed a special non-Hausdorff topology, where $\mathcal{O}$ represents a null preference point, a state of total indifferent to all candidates. However, this topology was quickly rejected by these authors because "it violated Hausdorff's separation axiom" (p. 134). Jones et al. (2003) seriously took up this idea and explored the existence of Chichilnisky maps when individual choices and/or the social outcome are allowed to have null preference $\mathcal{O}$. They were able to derive possibility results for the case of preference space modeled by S^n, where Resolution Theorem states that no Chichilnisky map could exist.

In this paper, we will describe a construction to insert a null preference $\mathcal{O}$ into any preference space P modeled as finite CW complexes rather than S^n. This new space $\tilde{P}$ is no longer a CW

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complex, but nonetheless $\tilde{P}$ is contractible. As contractibility is central to the Resolution Theorem, we will investigate here how a generalized construction of the "null closure" for any preference space P will affect the existence of Chichilnisky aggregation maps, and hence the resolution of topological social choice paradox.

After applying some tools from algebraic topology, we will be able to specify an output for profiles which contain votes that all "cancel out", and thus the problem is essentially reduced to the contractible case. Along the way, we will see that aggregation maps allowing individuals to have a null preference are not viable. We will also see that the existence of profiles that result in a "tie" is equivalent to a preference space being non-contractible. Finally, in the special case of $P = S^n$, we will show that the space of tie profiles is measure zero.

2. Contractibility and null preference in topological social choice

Let (X, τ) be any topological space. Define the space $(\tilde{X}, \tilde{\tau})$ by $\tilde{X} = X \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a new point, endowed with the topology $\tilde{\tau} = \tau \cup \{\tilde{X}\}$. That is, the only open neighborhood of $\mathcal{O}$ is the entire space. Intuitively, this means that the point $\mathcal{O}$ is "infinitely near" every other point of $\tilde{P}$. More precisely, the space $\tilde{X}$ is non-Hausdorff, and $\mathcal{O}$ is not separated from any other point in the sense that no neighborhood of $\mathcal{O}$ excludes any other point, and moreover every closed set in $\tilde{P}$ contains the point $\mathcal{O}$. This new space, which essentially follows the construction for S^n described in Le Breton and Uriarte (1990) and explored in length in Jones et al. (2003), has several useful properties, notably the fact that it has a non-empty specialization pre-order merely due to $\mathcal{O}$ as a common specialization of every point of X.

Lemma 2.1. *For any topological space X, the space $\tilde{X}$ is contractible.*

Proof. Define the map $F : \tilde{X} \times [0, 1] \to \tilde{X}$ by

$$F(x, t) = \begin{cases} x & \text{if } t < 1 \\ \mathcal{O} & \text{if } t = 1. \end{cases}$$

To see why this is continuous, note that there are two cases for an open set U in the image space. First, if $\mathcal{O} \notin U$, then the preimage is precisely $U \times [0, 1)$, which is open. The only remaining open subset of the image space is the entire range, which has the entire domain as preimage. Hence this is a continuous map, which defines a homotopy from the identity map to a constant map. □

Definition 2.2. Define the *null closure* of X to be the space $\tilde{X}$ obtained via the above construction.

In general, extending the preference space by adding a null preference requires the addition of one point and some open set (s) to the topology. Besides defining $\tilde{\tau}$ as discussed in the last subsection, the only other symmetrical way to define the topology on $P \cup \{\mathcal{O}\}$ would be to make $\mathcal{O}$ an isolated point, i.e. to make $\{\mathcal{O}\}$ an open set. However, this is then a non-contractible CW complex, for which the Resolution Theorem states that there are no Chichilnisky maps.

Le Breton and Uriarte (1990) "rejected" (p. 136) the null closure topology for $S^n \cup \{\mathcal{O}\}$ on the grounds that it does not satisfy the Hausdorff separation axiom. However, we think Hausdorff separability (which essentially turns a preference space into a metric space) is way too strong a requirement for social choice theories. We note that the null closure topology on $S^n \cup \{\mathcal{O}\}$, as the space of admissible preferences and/or social outcomes, only perturbed the Hausdorff topology on S^n "slightly", by having $\mathcal{O}$ as the only and common specialization point to all points of S^n—this is the only source of non-empty specialization order (and hence non-T_1 separability which precedes Hausdorff separability). This null closure construction augments the space of social outcomes to allow for Chichilnisky maps on any preference space, and has useful interpretations in applications, so we do not find this lack of Hausdorff separation to be a significant flaw. All Chichilnisky maps $P^n \to P$ are still Chichilnisky for $P^n \to \tilde{P}$.

3. Continuity and the null closure

With the construction of $\tilde{P}$, one next asks where it is appropriate to use $\tilde{P}$ or P. In Theorem 3.3, we use $\tilde{P}$ only as the range space of the aggregation map, and not as the space of possible preferences. It turns out that the choice made in Theorem 3.3 is the only one which allows for nontrivial aggregation maps. To show this, we take inspiration from Jones et al. (2003), where similar results are shown for the special case of $P = S^n$. We will follow a similar proof method, except that instead of using the metric structure on S^n, we will only need that the CW complex P is Hausdorff.

Proposition 3.1. *Let $f : (\tilde{P})^k \to \tilde{P}$ be continuous. Given any individual j and any profile $\mathbf{p}_{-j}$ for the remaining $k - 1$ voters, define the component map $\iota_j = f(\mathbf{p}_{-j}, \cdot) : \tilde{P} \to \tilde{P}$. Either $\iota_j(\mathcal{O}) = \mathcal{O}$ or else ι_j is constant.*

Proof. Consider $q = f(p_{-j}, \mathcal{O})$ and assume $q \neq \mathcal{O}$. Let $U \subseteq P$ be any open neighborhood of q. The preimage $(\iota_j)^{-1}(U)$ contains $\mathcal{O}$ and is open in $\tilde{P}$, and thus must be equal to all of $\tilde{P}$. Therefore $\mathrm{Im}(\iota_j) \subseteq U$.

Since U is arbitrary, we must have that $\mathrm{Im}(\iota_j)$ is contained in every open set containing q. But P is Hausdorff, so for any $p \in P$ with $p \neq q$, there will be some neighborhood of q which does not contain p. Hence the above intersection will be a singleton set $\{q\}$, and so ι_j is constant. □

Proposition 3.2. *The only continuous maps $f : (\tilde{P})^k \to P$ are the constant maps.*

Proof. First, note that $f(\bar{\mathcal{O}}) \in P$, where $\bar{\mathcal{O}}$ refers to the profile $(\mathcal{O}, \dots, \mathcal{O}) \in (\tilde{P})^k$. Hence for any open neighborhood of $f(\bar{\mathcal{O}})$, the preimage is an open subset of $(\tilde{P})^k$ containing $\bar{\mathcal{O}}$.

It is immediate from the definitions that the only open subset of $(\tilde{P})^k$ containing $\bar{\mathcal{O}}$ is the entire space.

Since the preimage of any neighborhood U of $f(\bar{\mathcal{O}})$ contains all of $(\tilde{P}^k)$, we have $\mathrm{Im}(f) \subseteq U$, where $U \subseteq P$ was arbitrary. By the same intersection argument as above, we have $\mathrm{Im}(f) = \{f(\mathcal{O})\}$, hence f is constant. □

To summarize, if we allow individual preferences to be null, each individual either forces the null outcome with their null vote, or has no effect whatsoever. These two results show that it is only reasonable to use the null closure in the range, i.e. the space of social preferences. In particular, no voter may choose the null preference, but the outcome may be the null preference.

Now, we may state precisely our extension of the Resolution Theorem.

Theorem 3.3. *Let P be a preference space realized as a parafinite CW complex, and $\tilde{P}$ the null closure. Then for all $k \geq 1$ there exists a Chichilnisky map $\Phi : P^k \to \tilde{P}$ such that the preimage of $\mathcal{O}$ is empty if and only if P is contractible.*

The proof is given in Appendix A. For a Chichilnisky map $\Phi : P^k \to \tilde{P}$, we will call $\Phi^{-1}(\{\mathcal{O}\})$ the set of *tie profiles*. The main idea of the proof, of course, is to construct a Chichilnisky map $P^k \to \tilde{P}$ for any preference space P and any $k \geq 1$. Our construction is similar to that in Chichilnisky and Heal (1983), with an additional step to find tie profiles to send to $\mathcal{O}$. As previously mentioned, this depends on both the space P and on the map Φ. We note that the contractible case of Theorem 3.3 is an immediate consequence of the Resolution Theorem.

The key step in Chichilnisky's proof, and in our generalization, involves using a retraction map from the convex hull of the preference space to the preference space itself. In the case of a non-contractible preference space, no such retraction exists, by a simple extension of the "no-retraction" theorem (Corollary 2.15 in Hatcher, 2001). The main obstruction is that each "hole" in the space, corresponding to a generator of a homotopy group, has no "central" point. That is, there is no consistent way to deal with a preference profile which is balanced around the hole, which we have referred to as a tie profile. In the case of finitely many candidates, this would correspond to two candidates with an equal number of votes. The simplest noncontractible example in the topological case is voting on the unit circle with diametrically opposed votes (with suitable aggregation function). If a single point is removed from the unit circle, it is contractible, in which case Chichilnisky's original result applies.

The construction of the null closure $\tilde{P}$ from P introduces the necessary central point to deal with the tie profiles, and admits a selection process which acknowledges the possibility of a null preference. This point $\mathcal{O}$ is essentially placeholder for inconclusive elections. Next, we explore a prototypical situation with $P = S^n$, and in particular, we will see in Proposition 4.3 that in this model the set of tie profiles is measure zero.

4. A Chichilnisky map for the null closure of spheres

We consider the case where the space of preferences is the n-sphere $P = S^n$. We recall the following aggregation rule $f : (S^n)^k \to \tilde{S^n}$, which has been considered before, for example in Jones et al. (2003). Let v_i represents the unit vector in $\mathbb{R}^{n+1}$ corresponding to the ith voter's choice. Then define f as follows:

$$f(v_1, v_2, \ldots, v_k) = \begin{cases} \dfrac{1}{\| \sum_{i=1}^{k} v_i \|} \displaystyle\sum_{i=1}^{k} v_i & \text{if } v_1 + \cdots + v_k \neq 0 \\ \mathcal{O} & \text{else.} \end{cases} \tag{4.1}$$

One can see that this provides a unit vector in $\mathbb{R}^{n+1}$, i.e. an element of S^n, except in the case when the vectors sum to zero, in which case we define the output to be the null preference $\mathcal{O}$. This is a natural choice of an aggregation map for $(S^n)^k \to \tilde{S^n}$. First, we show that it is a Chichilnisky map (cf. Le Breton and Uriarte, 1990, p. 134).

Proposition 4.2. *The social aggregation rule $f : (S^n)^k \to \tilde{S^n}$ defined in Eq. (4.1) satisfies anonymity, unanimity, and continuity.*

Proof. Anonymity and unanimity are immediate. We will now show that f is continuous.

We will first decompose f into two maps:

$$f : (S^n)^k \xrightarrow{\sigma} \mathbb{R}^{n+1} \xrightarrow{\pi} \tilde{S^n}.$$

Define $\sigma(v_1, \ldots, v_k) = v_1 + \cdots + v_k$ to be the vector sum in $\mathbb{R}^{n+1}$. This is of course continuous, as it is just a restriction of vector addition in k copies of $\mathbb{R}^{n+1}$. The map π is defined as follows:

$$\pi(x) = \begin{cases} x/\|x\| & \text{if } x \neq 0 \\ \mathcal{O} & \text{else.} \end{cases}$$

This is also easily seen to be continuous. The only open set in $\tilde{S^n}$ containing $\mathcal{O}$ is the entire space whose preimage will be all of $\mathbb{R}^{n+1}$, which is open. Any other open subset will also be an open subset of S^n, whose preimage will be an open cone in $\mathbb{R}^{n+1}$. We conclude that f is a composition of continuous maps, and so is itself continuous. □

In applications, it is certainly undesirable to have a null preference as the outcome of a voting situation. We will show that in the case considered here, namely the aggregation rule f defined above, the set of tie profiles has measure zero.

We will use the standard spherical measure on S^n by defining the measure of a set $A \subseteq S^n$, denoted $\mu(A)$, to be the Lebesgue measure for $\mathbb{R}^{n+1}$ of the following set:

$$A' = \{x \in \mathbb{R}^{n+1} \;:\; \exists a \in A \text{ such that } a = x/\|x\| \text{ and } \|x\| \leq 1\}$$

The set $A \subset S^n$ is defined to be measurable if $A' \subset \mathbb{R}^{n+1}$ is measurable. For the sphere $S^n \subseteq \mathbb{R}^{n+1}$, this measure is simply a scaled version of the n-dimensional surface area measured by the appropriate integral.

Proposition 4.3. *Using $f : (S^n)^k \to \tilde{S^n}$ as defined in Eq. (4.1), we have $\mu\big(f^{-1}(\{\mathcal{O}\})\big) = 0$.*

Proof. If $k = 1$, the preimage of $\mathcal{O}$ is empty. Consider now the case of $k \geq 2$. Define the map $g : (S^n)^k \to \mathbb{R}^{n+1}$ by

$$g(v_1, \ldots, v_k) = \|v_1 + \cdots + v_{k-1}\|.$$

This is a smooth map from a nk-dimensional smooth manifold to a 1-dimensional smooth manifold. Note that g is independent of v_k. Consider the following set $V \subseteq (\mathbb{R}^{n+1})^k$:

$$V := g^{-1}\big(\{1\}\big).$$

Let $U \subseteq (S^n)^k$ be defined as follows:

$$U = \{(v_1, \ldots, v_k) : v_1 + \cdots + v_k = 0\}.$$

By definition, $U = f^{-1}(\{\mathcal{O}\})$. It is clear that $U \subseteq V$, since the last term v_k must be able to cancel out the first $k-1$ terms. Because 1 is a regular value of g (that is, the derivative of g is nonzero at all preimages of 1), it is a standard fact from differential topology that V is a submanifold of $(S^n)^k$ of codimension 1 (Corollary 5.14 of Lee, 2013), and thus a measure zero set (Corollary 6.12 of Lee, 2013). But $U \subseteq V$, so $U = f^{-1}(\{\mathcal{O}\})$ has measure zero as well. □

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Appendix A. Proof of Theorem 3.3

We follow a similar approach to Chichilnisky and Heal in Chichilnisky and Heal (1983). We will start by building a contractible space from the preference space P, and then adapt the method from their original proof to this new space.

A.1. Building the appropriate contractible space

Lemma A.1. *The null closure $\tilde{S^n}$ of the n-dimensional sphere is a retract of the null closure of the closed $n+1$-dimensional disk $\tilde{\mathbb{D}}^{n+1}$.*

Proof. Place $\mathbb{D}^{n+1}$ in $\mathbb{R}^{n+1}$ in the standard manner. Define the retraction map $R : \tilde{\mathbb{D}}^{n+1} \to \tilde{S}^n$ as follows:

$$R(x) = \begin{cases} x/\|x\| & \text{if } x \neq 0 \\ \mathcal{O} & \text{if } x = 0. \end{cases}$$

It is straightforward to show that R is continuous, and hence a retraction. □

Now, we will build a convex hull for P using Lemma A.1. The convex hull is constructed in two steps. First, we will build a "contractible hull" inductively. Let $P_0 = P$. First, recall that $\pi_1(P)$ is finitely generated by loops contained in the 1-skeleton of P. Hence we may augment the 2-skeleton of P by gluing in finitely many 2-dimensional disks to these generators; call the resulting space P_1, and the embedding $i_0 : P_0 \hookrightarrow P_1$. This space is simply connected and all homology groups are finitely generated, so all homotopy groups are finitely-generated by Theorem B.4. In particular, the generators of $\pi_2(P_1)$ are contained in the 2-skeleton by Theorem B.5.

Now, inductively build P_n with $i_{n-1} : P_{n-1} \hookrightarrow P_n$ for $n = 2, 3, \ldots$ using the above procedure. Note that $\pi_k(P_i)$ is finitely generated for all k, so each step only involves gluing in finitely many copies of the n-disk. At each step, we have a parafinite CW complex P_n such that $\pi_k(P_n) = 0$ for $k = 1, \ldots, n$. Because the construction of P_n from P_{n-1} only adds to the $n+1$-skeleton finitely many cells which are attached to the n-skeleton, the directed system

$$P_0 \xrightarrow{i_0} P_1 \xrightarrow{i_1} \cdots \xrightarrow{i_n} P_{n+1} \xrightarrow{i_{n+1}} \cdots$$

has a limit, $\lim_{\rightarrow} P_i = P_\infty$, which will be a parafinite CW complex. Note that since $\pi_n(P_n) = 0$, and P_n is the $n+1$-skeleton of P_∞, the long exact sequence for homotopy groups (Theorem 4.3 of Hatcher, 2001) implies $\pi_n(P_\infty) = 0$ for all n. Hence P_∞ is contractible by Theorem B.3.

A.2. Building the averaging map

Define a retraction map $r_1 : P_\infty \to \tilde{P}$ using Lemma A.1. Points in $\tilde{P} \cap P_\infty$ will be sent to themselves. For the new disks in P_∞, the map R from Lemma A.1 may be applied, where the range will be the boundary of the disk, as well as the zero point of $\tilde{P}$. Next, we may take the convex hull $k(P_\infty)$.

Precisely as in the original proof in Chichilnisky and Heal (1983), we may define a retraction $r_2 : k(P_\infty) \to P_\infty$ by extending the identity map. Now, define the map ϕ by averaging the inputs, which is always possible since $k(P_\infty)$ is a convex space:

$$\phi(p_1, \ldots, p_k) = \frac{1}{k}\sum_{i=1}^{k} p_i.$$

The addition here is by vector addition using an implied embedding into Euclidean space. See the proof of the Resolution Theorem in Chichilnisky and Heal (1983) for further explanation. Now, define the map $\Phi := r_1 \circ r_2 \circ \phi$. The map Φ acts as follows:

$$\Phi : P^k \xrightarrow{\phi} k(P_\infty) \xrightarrow{r_2} P_\infty \xrightarrow{r_1} \tilde{P}.$$

The first map ϕ computes the average of the inputs, and this average must exist within $k(P_\infty)$. The map r_2 retracts the convex hull $k(P_\infty)$ onto the space P_∞, and the map r_1 sends null preferences to the point $\mathcal{O}$, and other valid preferences to their images in P.

Continuity of Φ follows since all maps involved are continuous. Anonymity is immediate from symmetry of ϕ. To see why unanimity holds, fix some $p \in P$ and consider $\Phi(p, \ldots, p) = r_1 \circ r_2 \circ \phi(p, \ldots, p)$. Obviously ϕ will return $p \in P$, and r_1 and r_2 are both retraction maps (aside from the case where r_1 takes in a zero point, which does not occur here), and so they restrict to the identity on P. Hence $\Phi(p, \ldots, p) = p$.

Notice that in the case that P is already contractible, we will have that $P_\infty = P$, and Φ will be the same map as in the original proof of the Resolution Theorem, so the preimage of $\mathcal{O}$ will be empty.

We have defined an aggregation rule on P which is continuous, unanimous, and anonymous, which defines our Chichilnisky map $\Phi : P^k \to \tilde{P}$. □

Appendix B. Background on topology and topological social choice

Definition B.1. A topological space (X, τ) is said to be *Hausdorff* if for any pair of distinct points $x, y \in X$, there exist open sets $U, V \in \tau$ such that $x \in U, y \in V$, and $U \cap V = \emptyset$.

Two of the most important topological spaces, which are both finite CW complexes, are the sphere:

$$S^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

and the (closed) disk:

$$\mathbb{D}^n = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}.$$

The boundary of a closed n-disk is homeomorphic to S^{n-1}.

Let X be a path-connected topological space. The nth homotopy group, denoted $\pi_n(X)$, is the group of continuous maps $S^n \to X$ based at a fixed point $x \in X$, up to homotopy. See Hatcher (2001) § 4.1 for a full discussion. Intuitively, $\pi_n(X)$ characterizes and codifies the $n+1$-dimensional holes in X. If there exists a finite list of elements of $\pi_n(X)$ which generate all other elements of the group by finite combinations, then $\pi_n(X)$ is said to be *finitely generated*. If the fundamental group $\pi_1(X)$ is trivial, the space is said to be *simply-connected*. Next, we have one of our most fundamental definitions:

Definition B.2. A space X is said to be *contractible* if there exists a continuous map $F : [0, 1] \times X \to X$ and some point $x_0 \in X$ such that $F(0, x) = x$ and $F(1, x) = x_0$.

That is, the identity map on X can be continuously deformed (via *homotopy*) to a constant map. Intuitively, this means the space can be continuously condensed down to a point. For example, a solid ball is contractible, but a (hollow) sphere is not.

Next, we will recall the definition of a parafinite CW complex; for more details, see Hatcher (2001) Chapter 0. A CW complex X is a Hausdorff space which is constructed inductively as follows. Starting with a discrete set of points X^0, one builds X^n from X^{n-1} by attaching a collection of n-disks and specifying where the boundary of the n-disk is glued. A CW complex is said to be parafinite if there are only finitely many cells of each dimension (but there may be infinitely many cells in total).

The following theorem is a very special case of a theorem of Whitehead; for a more detailed statement and proof, see Hatcher (2001), Theorem 4.5.

Theorem B.3. *If a CW complex X has $\pi_n(X) = 0$ for all n, then X is contractible.*

A collection of groups related to the homotopy groups, but easier to work with, are the homology groups, denoted $H_n(X)$. One tool for computing them, known as cellular homology, gives as an immediate result that the homology groups of a parafinite CW complex are finitely generated (see Hatcher, 2001, Chapter 2.2). Further, any parafinite CW complex will have finitely generated fundamental group.

A rather technical result which we will need gives a sufficient condition for a CW complex to have all homotopy groups be finitely generated. For the full statement and proof, see Hatcher (2004), Theorem 5.7. We state below the special case needed in this paper:

Theorem B.4. *If a CW complex X is simply connected, then $\pi_n(X)$ is finitely generated for all n if and only if $H_n(X)$ is finitely generated for all n.*

In particular, since we are dealing with parafinite CW complexes X, we know that $\pi_1(X)$ is finitely generated. And if $\pi_1(X) = 0$, then $\pi_n(X)$ is finitely generated for all $n \geq 1$. The Cellular Approximation Theorem allows us to assume that generators for $\pi_n(X)$ are contained in the n-skeleton X^n of a CW complex X. In particular:

Theorem B.5. *Every map $f : S^n \to X$ is homotopic to a map whose image is contained in the n-skeleton of X.*

We consider a space P of possible preference profiles among which voters decide. The space P of preferences is assumed to be a parafinite CW complex. This restriction guides much of the mathematical theory. A social aggregation rule (for k voters) is defined to be a map $f : P^k \to P$ which gives a social preference $f(p_1, \ldots, p_k) \in P$ for each profile $(p_1, \ldots, p_k) \in P^k$ of k individual preferences.

Definition B.6. We say f is a *Chichilnisky map* if it satisfies the following:

- anonymity: if σ is a permutation of $\{1, \ldots, k\}$, then for all voting profiles $(p_1, \ldots, p_k)$, we have

  $$f(p_1, \ldots, p_k) = f(p_{\sigma(1)}, \ldots, p_{\sigma(k)})$$
- unanimity: for all $p \in P$ we have $f(p, \ldots, p) = p$
- continuity: f is a continuous map.

A fundamental theorem in the theory of topological social choice is the Resolution Theorem, which gives a necessary and sufficient condition on the preference space P for the existence of a Chichilnisky map. It was first proven in Eckmann (1954) by Eckmann in slightly more generality; however, we will focus on the formulation and proof method of Chichilnisky in Chichilnisky and Heal (1983).

Theorem B.7 (*Chichilnisky and Heal, 1983, Resolution Theorem*)**.** *Let P be a preference space, realized as a parafinite CW complex. There exists for all natural numbers k a Chichilnisky map $f : P^k \to P$ if and only if P is contractible.*

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