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Abstract

On a statistical manifold (M, g, ∇), the Riemannian metric g is coupled to an (torsion-free) affine connection ∇, such that ∇g is totally symmetric; $\{\nabla, g\}$ is said to form "Codazzi coupling". This leads ∇^*, the g -conjugate of ∇, to have same torsion as that of ∇. In this paper, we investigate how statistical structure interacts with L in an almost Hermitian and almost para-Hermitian manifold (M, g, L), where L denotes, respectively, an almost complex structure J with $J^2 = -\mathrm{id}$ or an almost para-complex structure K with $K^2 = \mathrm{id}$. Starting with ∇^L, the L -conjugate of ∇, we investigate the interaction of (generally torsion-admitting) ∇ with L, and derive a necessary and sufficient condition (called "Torsion Balancing" condition) for L to be integrable, hence making (M, g, L) (para-)Hermitian, and for ∇ to be (para-)holomorphic. We further derive that ∇^L is (para-)holomorphic if and only if ∇ is, and that ∇^* is (para-)holomorphic if and only if ∇ is (para-)holomorphic and Codazzi coupled to g. Our investigations provide concise conditions to extend statistical manifolds to (para-)Hermitian manifolds.

(Para-)Holomorphic Connections for Information Geometry

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Abstract. On a statistical manifold (M, g, ∇), the Riemannian metric g is coupled to an (torsion-free) affine connection ∇, such that ∇g is totally symmetric; $\{\nabla, g\}$ is said to form "Codazzi coupling". This leads ∇^*, the g -conjugate of ∇, to have same torsion as that of ∇. In this paper, we investigate how statistical structure interacts with L in an almost Hermitian and almost para-Hermitian manifold (M, g, L), where L denotes, respectively, an almost complex structure J with $J^2 = -\text{id}$ or an almost para-complex structure K with $K^2 = \text{id}$. Starting with ∇^L, the L -conjugate of ∇, we investigate the interaction of (generally torsion-admitting) ∇ with L, and derive a necessary and sufficient condition (called "Torsion Balancing" condition) for L to be integrable, hence making (M, g, L) (para-)Hermitian, and for ∇ to be (para-)holomorphic. We further derive that ∇^L is (para-)holomorphic if and only if ∇ is, and that ∇^* is (para-)holomorphic if and only if ∇ is (para-)holomorphic and Codazzi coupled to g. Our investigations provide concise conditions to extend statistical manifolds to (para-)Hermitian manifolds. AQ1

1 Introduction

On the tangent bundle TM of a differentiable manifold M, one can introduce two separate structures: affine connection ∇ and pseudo-Riemannian metric g. A manifold M equipped with a g and a torsion-free connection ∇ is called a *statistical manifold* if (g, ∇) is Codazzi-coupled [Lau87]. This is the setting of "classical" information geometry, where the (g, ∇) pair arises from a general construction of divergence ("contrast") functions. To accommodate for torsions in affine connections, the concept of pre-contrast functions was introduced [HM11]. Codazzi coupling has been traditionally studied by affine geometers [NS94,Sim00]. The robustness of Codazzi coupling was investigated by perturbing both the metric and the affine connection [SSS09] and by its interaction with other transformations of connection [TZ16]. Below, we provide a succinct overview.

1.1 *g*-conjugate Connection, Cubic Form, and Codazzi Coupling

Given the pair (g, ∇), we construct the $(0, 3)$-tensor C by

$$C(X, Y, Z) := (\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y). \quad (1)$$

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The tensor C is sometimes referred to as the *cubic form* associated to the pair (∇, g). When $C = 0$, we say g is parallel under ∇.

Given the pair (g, ∇), we can also construct ∇^*, called g -conjugate connection, by

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y). \tag{2}$$

It can be checked easily that (i) ∇^* is indeed a connection and (ii) g-conjugation of a connection is involutive, i.e., $(\nabla^*)^* = \nabla$.

These two constructions from an arbitrary (g, ∇) pair are related via

$$C(X,Y,Z) = g(X, (\nabla^* - \nabla)_Z Y), \tag{3}$$

so that

$$C^*(X,Y,Z) := (\nabla^*_Z g)(X,Y) = -C(X,Y,Z).$$

Therefore $C(X,Y,Z) = C^*(X,Y,Z) = 0$ if and only if $\nabla^* = \nabla$, that is, ∇ is g-self-conjugate. A connection is both g -self-conjugate and torsion-free defines what is called the Levi-Civita connection ∇^{LC} associated to g.

Simple calculation reveals that

$$\begin{aligned} C(X,Y,Z) - C(Z,Y,X) &= (\nabla_Z g)(X,Y) - (\nabla_X g)(Z,Y), \\ C(X,Y,Z) - C(X,Z,Y) &= g(X, T^{\nabla^*}(Z,Y) - T^{\nabla}(Z,Y)), \end{aligned} \tag{4}$$

where T^∇ denotes the torsion of ∇

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Note that $C(X,Y,Z) = C(Y,X,Z)$ always holds, due to $g(X,Y) = g(Y,X)$. Therefore, imposing either of the following is equivalent:

1. $C(X,Y,Z) = C(Z,Y,X)$,
2. $C(X,Y,Z) = C(X,Z,Y)$;

this is because either (i) or (ii) will make C totally symmetric in all of its indices. In the case of (i), we say that g and ∇ are *Codazzi-coupled*:

$$(\nabla_Z g)(X,Y) = (\nabla_X g)(Z,Y). \tag{5}$$

In the case of (ii), ∇ and ∇^* have same torsion. These well-known facts are summarized in the following Lemma.

Lemma 1. *Let g be a pseudo-Riemannian metric, ∇ an arbitrary affine connection, and ∇^* be the g-conjugate connection of ∇. Then the following statements are equivalent:*

1. *(∇, g) is Codazzi-coupled;*
2. *(∇^*, g) is Codazzi-coupled;*
3. *C is totally symmetric;*
4. *C^* is totally symmetric;*
5. *$T^\nabla = T^{\nabla^*}$.*

In the above case, (g, ∇, ∇^*) is called a *Codazzi triple*. Codazzi-coupling between g and ∇ or, equivalently, the existence of Codazzi triple (g, ∇, ∇^*) is the key feature of a statistical manifold. In "quantum" information geometry, ∇ is allowed to carry torsion, and [Mat13] introduced *Statistical Manifold Admitting Torsion (SMAT)* as a manifold (M, g, ∇) satisfying

$$(\nabla_Y g)(X, Z) - (\nabla_X g)(Y, Z) = g(T^\nabla(X, Y), Z).$$

Note that ∇^* is torsion-free if and only if (M, g, ∇) is a SMAT. However, in a SMAT, neither ∇ nor ∇^* is Codazzi coupled to g; the deviation from Codazzi coupling is measured by the torsion T^∇ of ∇.

2 Structure of TM Arising from L

A tangent bundle isomorphism L may induce a splitting of TM, corresponding to the eigenbundles associated with the eigenvalues of L. How the action of an arbitrary connection ∇ respects such splitting is the focus of our current paper.

2.1 Splitting of TM by L

For a smooth manifold M, an isomorphism L of the tangent bundle TM is a smooth section of the bundle $\mathrm{End}(TM)$ such that it is invertible everywhere. By definition, L is called an *almost complex structure* if $L^2 = -\mathrm{id}$, or an *almost para-complex structure* if $L^2 = \mathrm{id}$ and the multiplicities of the eigenvalues ± 1 are equal. We will use J and K to denote almost complex structures and almost para-complex structures, respectively, and use L when these two structures can be treated in a unified way. It is clear from our definition that such structures exist only when M is of even dimension.

Denote eigenvalues of L as $\pm\alpha$, where $\alpha = 1$ for $L = K$ and $\alpha = i$ for $L = J$, respectively. Following the standard procedure, we (para-)complexify TM by tensoring with $\mathbb{C}$ or para-complex (also known as split-complex) field $\mathbb{D}$, and use $T^L M$ to denote the resulting $TM \otimes \mathbb{C}$ or $TM \otimes \mathbb{D}$, depending on the type of L. In analogy with standard notation in the complex case, let $T^{(1,0)}M$ and $T^{(0,1)}M$ be the eigenbundles of L corresponding to the eigenvalues $\pm\alpha$, i.e., at each point $p \in M$, the fiber is defined by

$$T^{(1,0)}(p) := \{X \in T_p^L M : L_p(X) = \alpha X\} \ ,$$
$$T^{(0,1)}(p) := \{X \in T_p^L M : L_p(X) = -\alpha X\} \ .$$

As sub-bundles of the (para-)complexified tangent bundle $T^L M$, $T^{(1,0)}M$ and $T^{(0,1)}M$ are distributions. A distribution is called a foliation if it is closed under the bracket $[\cdot, \cdot]$. We will refer to vectors to be of type $(1, 0)$ and $(0, 1)$ if they take values in $T^{(1,0)}M$ and $T^{(0,1)}M$ respectively. Moreover, define $\pi^{(1,0)}$ and $\pi^{(0,1)}$ to be the projections of a vector field to $T^{(1,0)}M$ and $T^{(0,1)}M$ respectively.

The Nijenhuis tensor N_L associated with L is defined as

$$N_L(X, Y) = -L^2[X, Y] + L[X, LY] + L[LX, Y] - [LX, LY]. \tag{6}$$

When $N_L = 0$, the operator L is said to be integrable. It is well-known that both $T^{(1,0)}M$ and $T^{(0,1)}M$ are foliations if and only if L is integrable, i.e., the integrability condition $N_L = 0$ is satisfied.

2.2 L-conjugate of ∇

Starting from a (not necessarily torsion-free) connection ∇ operating on sections of TM, we can apply an L-conjugate transformation to obtain a new connection $\nabla^L := L^{-1}\nabla L$, or

$$\nabla^L_X Y = L^{-1}(\nabla_X(LY)) \tag{7}$$

for any vector fields X and Y; here L^{-1} denotes the inverse isomorphism of L. It can be verified that indeed ∇^L is an affine connection.

Define a (1, 2)-tensor (vector-valued bilinear form) S via the expression

$$S(X,Y) = (\nabla_X L)Y - (\nabla_Y L)X, \tag{8}$$

where

$$(\nabla_X L)Y = \nabla_X(LY) - L(\nabla_X Y).$$

We say that L and ∇ are *Codazzi-coupled* if $S = 0$. The following is known.

Lemma 2 *(e.g., [SSS09]). Let ∇ be an affine connection, and let L be an arbitrary tangent bundle isomorphism. Then the following statements are equivalent:*

(i) (∇, L) is Codazzi-coupled.
(ii) $T^\nabla(X,Y) = T^{\nabla^L}(X,Y)$.
(iii) (∇^L, L^{-1}) is Codazzi-coupled.

Lemma 3. *For the special case of (para-)complex operators $L^2 = \pm\mathrm{id}$,*

1. $\nabla^L = \nabla^{L^{-1}}$, i.e., L-conjugate transformation is involutive, $(\nabla^L)^L = \nabla$.
2. (∇, L) is Codazzi-coupled if and only if (∇^L, L) is Codazzi-coupled.

As an affine connection, ∇ gives rise to a map

$$\nabla : \Omega^0(TM) \to \Omega^1(TM),$$

where $\Omega^i(TM)$ is the space of smooth i-forms with value in TM. We may extend this to a map

$$d^\nabla : \Omega^i(TM) \to \Omega^{i+1}(TM)$$

by

$$d^\nabla(\alpha \otimes v) = d\alpha \times v + (-1)^i \alpha \wedge \nabla v$$

for any i-form α and vector field v. In the case that ∇ is flat, then $(d^\nabla)^2 = 0$ and we get a chain complex whose cohomology is the de Rham cohomology twisted by the local system determined by ∇. Regarding L as an element of $\Omega^1(TM)$, it is easy to check using local coordinates that

$$(d^\nabla L)(X,Y) = (\nabla_X L)Y - (\nabla_Y L)X + LT^\nabla(X,Y). \tag{9}$$

Therefore, Codazzi coupling of ∇ and L can also be expressed as

$$(d^\nabla L)(X,Y) = T^\nabla(LX,Y). \tag{10}$$

2.3 Integrability of L

In [FZ17, Lemma 2.5] an expression for $N_L(X,Y)$ in terms of T^∇ has been derived assuming $S=0$. Using exactly the same procedure, we can write down $N_L(X,Y)$ for an arbitrary S.

Lemma 4. *Given a connection ∇ with torsion T^∇, the Nijenhuis tensor N_L of a (para-)complex operator L is given by*

$$\begin{aligned} N_L(X,Y) &= L^2T^\nabla(X,Y) - LT^\nabla(X,LY) - LT^\nabla(LX,Y) + T^\nabla(LX,LY) \\ &\quad + LS(X,Y) - L^{-1}S(LY,LX). \end{aligned}$$

Now, define θ to be

$$\theta(X,Y) = \frac{1}{2}(\nabla^L_X Y - \nabla_X Y) = \frac{1}{2}L^{-1}(\nabla_X L)Y. \tag{11}$$

with

$$L\theta(X,Y) + \theta(X,LY) = 0. \tag{12}$$

In particular, we see that

$$\frac{1}{2}L^{-1}(S(X,Y)) = \theta(X,Y) - \theta(Y,X),$$

and therefore, θ is symmetric if and only if L and ∇ are Codazzi-coupled. Introduce

$$\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^L),$$

which satisfies

$$\tilde{\nabla}L \equiv 0.$$

A connection with respect to which L is parallel is called *(para-)complex connection*, and in particular, such a connection preserves the decomposition $T^LM \cong T^{(1,0)}M \oplus T^{(0,1)}M$. So starting from any connection ∇, we can construct its conjugate ∇^L, the average of which is the (para-)complex connection $\tilde{\nabla}$. This situation mirrors the relationship between Levi-Civita connection and the pair of g-conjugate connections ∇, ∇^*. Note that we can also write $\nabla = \tilde{\nabla} - \theta$ and $\nabla^L = \tilde{\nabla} + \theta$, so the quantity θ measures the failure of both ∇ and ∇^L to be a (para-)complex connection.

3 (Para-)Holomorphicity of ∇ Associated to L

3.1 (Para-)Holomorphic Connections

The (para-)Dolbeault operator $\bar{\partial}$ for a given L on T^LM is defined as

$$\bar{\partial}_X Y = \frac{1}{4}\left([X,Y] - L^2[LX,LY] - L^{-1}[LX,Y] + L^{-1}[X,LY]\right) \tag{13}$$

for any vector fields X and Y. It can be checked easily that this expression is tensorial in X, that is $\bar{\partial}_{fX}Y = f\left(\bar{\partial}_X Y\right)$ and is a derivation. In the case when $L = J$, this defines the *holomorphic structure* on $T^{\mathbb{C}}M$ and locally defines the differentiation of vector fields of type $(1,0)$ with respect to the anti-holomorphic coordinates $\frac{\partial}{\partial \bar{z}^i}$. Similarly for *para-holomorphic structure* on $T^{\mathbb{D}}M$ when $L = K$.

From (13) we obtain that if X and Y are of the same type, then $\bar{\partial}_X Y = 0$. However, if $Y \in T^{(1,0)}M$ and $X \in T^{(0,1)}M$, then

$$\bar{\partial}_X Y = \pi^{(1,0)}\left[X, Y\right] \tag{14}$$

and similarly $\bar{\partial}_X Y = \pi^{(0,1)}\left[X, Y\right]$ if $Y \in T^{(0,1)}M$ and $X \in T^{(1,0)}M$. Equivalently, note that if $X \in T^{(1,0)}M$, then $\bar{\partial}X$ is a vector-valued 1-form, of type $(1,0)$ as a vector and type $(0,1)$ as a 1-form, and conversely if $X \in T^{(0,1)}M$.

Given a connection ∇ operating on $T^L M$, we can ask the question whether ∇ is compatible with $\bar{\partial}$. To understand this we may define an alternative operator $\bar{\partial}^{\nabla}$, which for $Y \in T^{(1,0)}M$ is defined as taking the $(0,1)$-part of the vector-valued 1-form ∇Y (and conversely on $T^{(0,1)}M$). This can be expressed as

$$\bar{\partial}^{\nabla}_X Y = \frac{1}{2}\left(\nabla_X Y - \nabla_{LX}\left(L^{-1}Y\right)\right) \tag{15}$$

for any vector fields X and Y in $T^L M$. Clearly, $\bar{\partial}^{\nabla}_X Y = 0$ if X and Y are of the same type and is just $\nabla_X Y$ if X and Y are of opposite type. On a (para-)holomorphic vector bundle, a connection is said to be *(para)-holomorphic* if these two Dolbeault operators coincide. We extend this notion to arbitrary connections on $T^L M \cong T^{(1,0)}M \oplus T^{(0,1)}M$ (that do not necessarily preserve $T^{(1,0)}M$ and $T^{(0,1)}M$) – we say a connection ∇ is *(para-)holomorphic* if $\bar{\partial}^{\nabla}_X Y = \bar{\partial}_X Y$ for any vector fields X and Y.

It can be readily shown that

Theorem 1. *∇^L is (para-)holomorphic if and only if ∇ is (para-)holomorphic.*

Theorem 2. *When ∇ is (para-)holomorphic, the quantity $\theta\left(X, Y\right)$ satisfies:*

$$L\theta\left(X, Y\right) = -\theta\left(X, LY\right) = -\theta\left(LX, Y\right) = L^{-1}\theta\left(LX, LY\right). \tag{16}$$

Theorem 2 shows that $\theta\left(X, Y\right)$ vanishes whenever X and Y are of different types. Moreover, if X and Y are both of type $(1,0)$, $\theta\left(X, Y\right)$ is of type $(0,1)$, and vice versa.

Using (13) and (15), we can also prove

Lemma 5. *Given an arbitrary connection ∇ and an L on a manifold, the connection ∇ is (para-)holomorphic if and only if*

$$S(X, Y) = T^{\nabla}(LX, Y) - LT^{\nabla}(X, Y) - \frac{1}{2}L^2 N_L(LX, Y). \tag{17}$$

From this, we prove the main theorem of our paper.

Theorem 3. *Given the an arbitrary pair* (∇, L) *on a manifold, the connection* ∇ *is (para-)holomorphic and* L *is integrable if and only if*

$$S(X,Y) = T^{\nabla}(LX,Y) - LT^{\nabla}(X,Y). \tag{18}$$

The significance of Theorem 3 is that this gives us a generalization of the Codazzi coupling condition for L that was used in [FZ17] in the case $T^{\nabla} = 0$. In fact, it follows immediately that if $T^{\nabla} = 0$ then Codazzi coupling of ∇ with L makes L integrable *and* makes ∇ (para-)holomorphic.

The condition (18) can be recast in another form to reveal its meaning:

Theorem 4. *Given* ∇ *and* L *on a manifold, then* ∇ *is (para-)holomorphic and* L *is integrable if and only if*

$$T^{\nabla}(LX,Y) = L(T^{\nabla^L}(X,Y)). \tag{19}$$

Theorem 4 shows that the (para-)holomorphicity condition on ∇ can be thought of as requiring "Torsion-Balancing" between ∇ and ∇^L.

3.2 Almost (Para-)Hermitian Structure

The compatibility condition between g and an almost (para-)complex structure $J(K)$ is well-known. We say that g is compatible with J if J is orthogonal, i.e.

$$g(JX,JY) = g(X,Y) \tag{20}$$

holds for any vector fields X and Y. Similarly we say that g is compatible with K if

$$g(KX,KY) = -g(X,Y) \tag{21}$$

is always satisfied, which implies that g must be of split signature. When expressed using L, (20) and (21) have the same form

$$g(X,LY) + g(LX,Y) = 0. \tag{22}$$

When specified in terms of compatible g and L, the manifold (M,g,L) is said to be almost (para-)Hermitian, and (para-)Hermitian manifold if L is integrable.

For any almost (para)-Hermitian manifold, we can define the 2-form $\omega(X,Y) = g(LX,Y)$, called the *fundamental form*, which turns out to satisfy $\omega(X,LY) + \omega(LX,Y) = 0$. The three structures, a pseudo-Riemannian metric g, a nondegenerate 2-form ω, and a tangent bundle isomorphism $L : TM \to TM$ forms a "compatible triple" such that given any two, the third one is uniquely specified; the triple is rigidly "interlocked".

It can be shown that for almost (para-)Hermitian manifolds,

$$(\nabla^L_X g)(LY,Z) + (\nabla_X g)(Y,LZ) = 0. \tag{23}$$

3.3 (Para-)Holomorphicity of ∇^*

We have seen in Theorem 1 that ∇ is (para-)holomorphic if and only if ∇^L is also (para-)holomorphic. We now investigate conditions under which ∇^* is also (para-)holomorphic whenever ∇ is.

Lemma 6. *Given arbitrary g and L on a manifold, with a (para-)holomorphic connection ∇. Then ∇^* is also (para-)holomorphic if and only if*

$$C(LX, Y, Z) = C(X, Y, LZ) \tag{24}$$

for any vector fields X, Y, Z. If moreover, g and L are compatible, i.e., (22) holds, then (24) is equivalent to

$$C(X, Y, Z) = g(\theta(Z, X), Y) + g(X, \theta(Z, Y)). \tag{25}$$

The condition that ∇^* is (para-)holomorphic is a very strong one as the theorem below shows.

Theorem 5. *Let ∇ be a (para-)holomorphic connection ∇ on an almost (para-) Hermitian manifold (M, g, L). Then, the connection $\tilde{\nabla} = \frac{1}{2}\left(\nabla + \nabla^L\right)$ is metric-compatible if and only if ∇^* is also (para-)holomorphic.*

In fact, since we already know that $\tilde{\nabla}$ is a (para-)complex connection, i.e. it preserves L, the condition of ∇^* being (para-)holomorphic is then equivalent to $\tilde{\nabla}$ being an almost (para-)Hermitian connection. Moreover, if we assume L to be integrable, since $\tilde{\nabla}$ is also (para-)holomorphic, we can conclude that when restricted to bundle $T^{(1,0)}M$, it must be equal to the (para-)Chern connection. In the theory of holomorphic vector bundles, Chern connection is the unique Hermitian holomorphic connection on a holomorphic vector bundle, and in particular on $T^{(1,0)}M$ on complex manifolds [Mor07]. In general, the Chern connection has torsion, however it is torsion-free on $T^{(1,0)}M$ if and only if (g, J) define a Kähler structure.

It is significant that if g is Codazzi-coupled to a (para-)holomorphic connection ∇, then ∇^* is (para-)holomorphic, and hence $\tilde{\nabla}$ is (para-)Hermitian.

Theorem 6. *Let (M, g, L) be a (para-)Hermitian manifold and let (∇, ∇^*, g) be a Codazzi triple. Then (∇^*, g) is (para-)holomorphic if and only if (∇, g) is (para-)holomorphic.*

This generalizes the results on a Codazzi-(para-)Kähler manifold [FZ17] which admit a pair of torsion-free connections to a (para-)Hermitian manifold which admits holomorphic connections with torsion. The Torsion-Balancing condition, while breaking the requirements of (para-)Kähler structure by possibly violating $d\omega = 0$, still preserves the integrability of L.

4 Summary and Discussions

(Para-)holomorphic connections have hardly been systematically studied in information geometry except in restricted setting of flat connections (see [Fur09]). Connections investigated in this paper are neither curvature-free nor torsion-free. We gave a necessary and sufficient condition("Torsion Balance") of a ∇ to be (para-)holomorphic in the presence of a (para-)complex structure L on the manifold. Given a (para-)holomorphic connection ∇, we then showed that (i) ∇^L, its L-conjugate, is also (para-)holomorphic; (ii) ∇^*, its g -conjugate, is (para-)holomorphic if and only if g and ∇ are Codazzi coupled. These concise characterizations allow us to enhance a statistical structure to a (para-)Hermitian structure, as well as understand the properties of L-conjugaty and g-conjugacy of a connection of a (para-)Hermitian manifold.

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