Results Math *Online First*

DOI 10.1007/s00025-017-0711-7

Results in Mathematics

Interaction of Codazzi Couplings with (Para-)Kähler Geometry

Teng Fei and Jun Zhang

Abstract. We study Codazzi couplings of an affine connection ∇ with a pseudo-Riemannian metric g, a nondegenerate 2-form ω, and a tangent bundle isomorphism L on smooth manifolds, as an extension of their parallelism under ∇. In the case that L is an almost complex or an almost para-complex structure and (g, ω, L) form a compatible triple, we show that Codazzi coupling of a torsion-free ∇ with any two of the three leads to its coupling with the remainder, which further gives rise to a (para-)Kähler structure on the manifold. This is what we call a *Codazzi-(para-)Kähler structure*; it is a natural generalization of special (para-)Kähler geometry, without requiring ∇ to be flat. In addition, we also prove a general result that g-conjugate, ω-conjugate, and L-gauge transformations of ∇, along with identity, form an involutive Abelian group. Hence a Codazzi-(para-)Kähler manifold admits a *pair* of torsion-free connections compatible with the (g, ω, L). Our results imply that any statistical manifold may admit a (para-)Kähler structure as long as one can find an L that is compatible to g and Codazzi coupled with ∇.

Mathematics Subject Classification. 32Q15, 32Q60, 53B05, 53B35, 53D05, 62B10.

Keywords. Codazzi coupling, conjugate connection, gauge transformation, Kähler structure, Para-Kähler structure, statistical manifold, Torsion.

1. Introduction

Let M be a smooth (real) manifold and ∇ be a torsion-free connection on it. In this paper, we would investigate the interaction of ∇ with three geometric structures on M, namely, a pseudo-Riemannian metric g, a nondegenerate 2-form ω, and a tangent bundle isomorphism $L : TM \to TM$, often forming a "compatible triple" together. The interaction of the compatible triple with ∇,

Published online: 05 July 2017

in terms of parallelism, is well understood, leading to integrability of L and of ω, and turning almost (para-)Hermitian structure to (para-)Kähler structure. Here, we investigate the interaction of ∇ with the compatible triple in terms of Codazzi coupling, a relaxation of parallelism.

The most important examples of the bundle isomorphism L are almost complex structures and almost para-complex structures. By definition, L is called an *almost complex structure* if $L^2 = -\mathrm{id}$. Analogously, L is known as an *almost para-complex structure* if $L^2 = \mathrm{id}$ and the multiplicities of the eigenvalues ± 1 are equal. We will use J and K to denote almost complex structures and almost para-complex structures, respectively, and use L when these two structures can be treated in a unified way. It is clear from our definition that such structures exist only when M is of even dimension.

The compatibility condition between a metric g and an almost (para-) complex structure $J(K)$ is well-known. We say that g is compatible with J if J is orthogonal, i.e.

$$g(JX, JY) = g(X, Y) \tag{1}$$

holds for any vector fields X and Y. Similarly we say that g is compatible with K if

$$g(KX, KY) = -g(X, Y) \tag{2}$$

is always satisfied, which implies that g must be of split signature. When expressed using L, (1) and (2) have the same form

$$g(X, LY) + g(LX, Y) = 0. \tag{3}$$

Hence a (0,2)-tensor ω can be defined

$$\omega(X, Y) = g(LX, Y), \tag{4}$$

and turns out to satisfy

$$\omega(X, LY) + \omega(LX, Y) = 0. \tag{5}$$

Of course, one can also start with ω and define $g(X, Y) = \omega(L^{-1}X, Y)$, then show that imposing compatibility of ω and L via (5) leads to the desired symmetry of g. Finally, given the knowledge of both g and ω, the bundle isomorphism L defined by (4) is uniquely determined, which satisfies (3), (5) and $L^2 = \pm id$. Whether L takes the form of J or K depends on whether (1) as opposed to (2) is to be satisfied.

In any case, the three objects g, ω and L form a compatible triple such that given any two, the third one is rigidly "interlocked". When specified in terms of compatible g and L, the manifold (M, g, L) is said to be almost (para-)Hermitian, and (para-)Hermitian manifold if L is integrable. On the other hand, when specified in terms of a nondegenerate 2-form ω, the manifold (M, ω) is said to be symplectic if we require ω to be closed. Amending (M, ω) with a (not necessarily integrable) L turns (M, ω, L) into an almost (para-) Kähler manifold. When we require both (i) an integrable L and (ii) a closed ω, then what we have on M is a *(para-)Kähler structure*.

Though the definition of a (para-)Kähler manifold does not involve any connections, it is the integrability conditions of L and ω that make the properties of ∇ relevant. A well-known result states that (M, g, L) is (para-)Kähler if and only if L is parallel under the Levi-Civita connection of g. In other words, there exists a torsion-free connection ∇ such that

$$\nabla g = 0, \quad \nabla L = 0.$$

Since the parallelism of L with respect to any torsion-free ∇ implies that L is integrable, a symplectic manifold (M, ω) can be enhanced to a (para-)Kähler manifold if any symplectic connection on M renders L parallel:

$$\nabla \omega = 0, \quad \nabla L = 0.$$

In this paper, we investigate integrability of L and of ω while g and L are not necessarily covariant constant with respect to ∇. The compatibility of ∇ with g or L is captured by the so-called Codazzi coupling, generalizing the aforementioned parallelism. Codazzi coupling of (∇, g) characterizes what is known to information geometers as *statistical structures* [11], and is widely studied in affine differential geometry [18,21]. As a special case, Hessian manifolds (for which ∇ is flat but not Levi-Civita) are the affine analogue of Kähler manifolds, see [3,20].

The contributions of our investigations include:

(a). Theorem 2.13 provides the structural result that the Kleinian group acts on an arbitrary affine connection by g-conjugation, ω-conjugation, and L-gauge transformation.

(b). Theorem 3.4 explains the relationship of Codazzi couplings of a torsion-free connection with a compatible triple. This enhances a compatible triple to a compatible "quadruple".

(c). Propositions 3.10 and 3.11 (along with Definition 3.9) show compatibility of a pair of connections with Kähler and para-Kähler structures. This result readily generalizes special Kähler geometry (where the connection is curvature-free) to Codazzi–Kähler geometry (where the connection need not be curvature-free). Special Kähler manifolds have abundant applications in string theory.

Though the derivations are standard (and even elementary), our paper fills the gap between Codazzi coupling and Kähler/para-Kähler geometry which have not been systematically attended to. The results are of interest with respect to affine differential geometry and the interdisciplinary area of Information Geometry, in which Codazzi coupling and conjugate connections play a key role.

The structure of the paper is as follows. In Sect. 2, we investigate Codazzi coupling of an arbitrary affine connection ∇ with (g, ω, L), and show how the respective Codazzi coupling results in torsion preservation upon conjugation of ∇ by g or by ω, or upon its gauge transform by L. We then prove a key result stating that g-conjugation, ω-conjugation, and L-gauge transformation (along with an identity operation) together act as the 4-element Klein group (Theorem 2.13). As a corollary, g-conjugation and L-gauge transformation are identical when ∇ is almost symplectic (Corollary 2.14). Along with our proof that Codazzi coupling of ∇ with L implies the integrability of L (Proposition 2.6), we arrive at our main result that Codazzi coupling of ∇ with both g and L gives rise to a (para-)Kähler manifold (Theorem 3.2). In Sect. 3, we introduce the notion of Codazzi-(para-)Kähler structure, as a (para-)Kähler structure with an additional "nice" affine connection ∇. Just as a (para-)Kähler manifold is an integrable structure of a compatible triple (g, ω, L), a Codazzi-(para-)Kähler manifold M is an integrable "compatible quadruple" (Definition 3.9); it admits a pair of torsion-free connections, both Codazzi-coupled with each member of the compatible triple (g, ω, L) (Propositions 3.10 and 3.11). Codazzi-(para-)Kähler manifold is a natural generalization of special (para-)Kähler geometry, without requiring ∇ (and hence its Codazzi-dual) to be flat. Essentially, Codazzi-(para-)Kähler structure is simultaneously a statistical structure and a (para-)Kähler structure (Corollary 3.6). In Sect. 4, we close our paper with some discussions of our results under the context of special Kähler geometry in theoretical physics and information geometry.

2. Codazzi Coupling, Conjugation, and Torsion

In this section, we investigate Codazzi couplings of an affine connection ∇ on a real manifold M with a pseudo-Riemannian metric g and a tangent bundle isomorphism $L : TM \to TM$. We prove that the Codazzi coupling between a torsion-free ∇ and a quadratic operator L leads to transversal foliations. Mirroring the study of these Codazzi couplings is the study the transformations of ∇ by g-conjugate, by ω-conjugate, and by L-gauge, which are all related to preservation of torsion of ∇. As a highlight, we show that these transformations generically are non-trivial elements of the four-element Kleinian group.

2.1. Codazzi Coupling of ∇ with L

For a smooth manifold M, an isomorphism L of the tangent bundle TM is a smooth section of the bundle End TM such that it is invertible everywhere. Starting from a (not necessarily torsion-free) connection ∇ on TM, we can apply an L-gauge transformation to obtain a new connection ∇^L defined by

$$\nabla^L_X Y = L^{-1}(\nabla_X(LY))$$

for any vector fields X and Y. It can be verified that indeed ∇^L is an affine connection. This is just the standard gauge group action on the space of affine connections.

Definition 2.1. We say that L and ∇ are *Codazzi-coupled* if the following identity holds

$$(\nabla_X L)Y = (\nabla_Y L)X, \tag{6}$$

where

$$(\nabla_X L)Y \equiv \nabla_X(LY) - L(\nabla_X Y).$$

We have the following characterization of Codazzi coupling:

Proposition 2.2. (e.g., [22]) *Let ∇ be an affine connection, and let L be a tangent bundle isomorphism. Then the following statements are equivalent:*

(a). (∇, L) *is Codazzi-coupled.*
(b). ∇ *and* ∇^L *have equal torsions.*
(c). (∇^L, L^{-1}) *is Codazzi-coupled.*

Proof. Recall the torsion tensor $T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, it is straightforward to obtain

$$L^{-1}((\nabla_X L)(Y) - (\nabla_Y L)(X)) = T^{\nabla^L}(X,Y) - T^\nabla(X,Y).$$

Hence (a) and (b) are equivalent. Furthermore, because $(\nabla^L)^L = \nabla$, (b) is equivalence to (c) on accounts of its equivalence to (a). □

One can easily prove the following for the special case of $L^2 = \pm\mathrm{id}$.

Corollary 2.3. *When L is either an almost complex structure J or an almost para-complex structure K,*

(a). $\nabla^L = \nabla^{L^{-1}}$, *i.e., L-gauge transformation is involutive,* $(\nabla^L)^L = \nabla$.
(b). (∇, L) *is Codazzi-coupled if and only if* (∇^L, L) *is Codazzi-coupled.*

There is another way to understand the Codazzi-coupling of ∇ and L. As an affine connection, ∇ gives rise to a map

$$\nabla : \Omega^0(TM) \to \Omega^1(TM),$$

where $\Omega^i(TM)$ is the space of smooth i-forms with value in TM. We may extend this to a map

$$\mathrm{d}^\nabla : \Omega^i(TM) \to \Omega^{i+1}(TM)$$

by

$$\mathrm{d}^\nabla(\alpha \otimes v) = \mathrm{d}\alpha \times v + (-1)^i \alpha \wedge \nabla v$$

for any i-form α and vector field v. In the case that ∇ is flat, then $(\mathrm{d}^\nabla)^2 = 0$ and we get a chain complex whose cohomology is the de Rham cohomology twisted by the local system determined by ∇.

Regarding L as an element of $\Omega^1(TM)$, it is easy to check using local coordinates that

$$(\mathrm{d}^\nabla L)(X,Y) = (\nabla_X L)Y - (\nabla_Y L)X + LT^\nabla(X,Y),$$

where the torsion tensor T^∇ is given by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Therefore Codazzi-coupling of ∇ and L is equivalent to

$$\mathrm{d}^{\nabla} L = LT^{\nabla}.$$

In particular it implies that L is d^{∇}-closed when ∇ is torsion-free.

2.2. Quadratic Operator and Transversal Foliations

Definition 2.4. (*Quadratic operator*) A tangent bundle isomorphism $L : TM \to TM$ is called a *quadratic operator* if it satisfies a real coefficient quadratic polynomial equation with distinct roots, i.e., there exists $\alpha \neq \beta \in \mathbb{C}$ such that $\alpha + \beta,\ \alpha\beta$ are real numbers and

$$L^2 - (\alpha + \beta)L + \alpha\beta \cdot \mathrm{id} = 0.$$

Note that L is an isomorphism, so $\alpha\beta \neq 0$. The most important examples of such operators are almost complex structures $J^2 = -\mathrm{id}$ and almost para-complex structures $K^2 = \mathrm{id}$ on TM.

Let E_α and E_β be the eigenbundles of L corresponding to the eigenvalues α and β respectively, i.e., at each point $p \in M$, the fiber is defined by

$$E_\lambda(p) := \{x \in T_pM : L_p(x) = \lambda x\} \quad \text{for } \lambda = \alpha, \beta.$$

As sub-bundles of the tangent bundle TM, E_α and E_β are distributions. We call $E_\alpha(E_\beta)$ a foliation if for any vector fields X, Y with value in $E_\alpha(E_\beta)$, so is their Lie bracket $[X, Y]$.

The Nijenhuis tensor N_L associated with L is defined as

$$N_L(X, Y) = -L^2[X, Y] + L[X, LY] + L[LX, Y] - [LX, LY].$$

When $N_L = 0$, the operator L is said to be *integrable*. It is well-known that both E_α and E_β are foliations if and only if L is integrable, i.e., the integrability condition $N_L = 0$ is satisfied. If α and β are not real, this makes sense only after we complexify TM by tensoring with $\mathbb{C}$.

Lemma 2.5. *Let L be a quadratic operator which is Codazzi-coupled to an affine connection ∇. Then*

$$N_L(X, Y) = L^2T^{\nabla}(X, Y) - LT^{\nabla}(X, LY) - LT^{\nabla}(LX, Y) + T^{\nabla}(LX, LY).$$

Proof. For any affine connection ∇, the condition

$$(\nabla_X L)Y = (\nabla_Y L)X$$

is equivalent to

$$\nabla_X(LY) - \nabla_Y(LX) - L[X, Y] = LT^{\nabla}(X, Y). \tag{7}$$

Substitute X and Y by LY and LX respectively, we get

$$\nabla_{LY}(L^2X) - \nabla_{LX}(L^2Y) - L[LY, LX] = LT^{\nabla}(LY, LX).$$

Using the assumption that $L^2 - (\alpha + \beta)L + \alpha\beta \cdot \mathrm{id} = 0$, we can rewrite it as

$$(\alpha + \beta - L)[LY, LX] - \alpha\beta(\nabla_{LY}X - \nabla_{LX}Y) = (L - \alpha - \beta)T^{\nabla}(LY, LX). \tag{8}$$

By computing $L(\alpha\beta(7) + (8))$, we get

$$\begin{aligned}\alpha\beta N_L(X,Y) &= \alpha\beta(-L^2[X,Y] + L[X,LY] + L[LX,Y] - [LX,LY]) \\ &= \alpha\beta(L^2T^\nabla(X,Y) - LT^\nabla(X,LY) - LT^\nabla(LX,Y) \\ &\quad + T^\nabla(LX,LY)).\end{aligned}$$

□

An immediate corollary of the above is that, if ∇ is torsion-free, then the Nijenhuis tensor N_L vanishes and we get two transversal foliations E_α and E_β. That is,

Proposition 2.6. *A quadratic operator L is integrable if it is Codazzi-coupled to a torsion-free connection ∇.*

Combining Proposition 2.6 with 2.2, we have

Proposition 2.7. *A quadratic operator L is integrable if there exists a torsion-free connection ∇ such that ∇^L is torsion-free.*

Remark 2.8. One wonders whether the converse of Proposition 2.6 is true or not. The answer is negative, because N_L also vanishes when the operator L satisfies, instead of (6), the following relation for any torsion-free connection ∇:

$$L(\nabla_X L)Y = (\nabla_{LX} L)Y; \tag{9}$$

see, e.g., [16, Lemma 11.4] for a proof for the almost complex case. In fact, because of the identity

$$0 = \nabla_X(L^2) = (\nabla_X L)L + L(\nabla_X L), \tag{10}$$

we see that (6) leads to

$$L(\nabla_X L)Y = -(\nabla_X L)(LY) = -(\nabla_{LY} L)X,$$

contrary to (9) by a sign. Therefore, (9), as a condition implying $N_L = 0$, cannot itself be derived from (6).

2.3. Codazzi Coupling of ∇ with g

Now we recall the Codazzi coupling of ∇ with a pseudo-Riemannian metric g on M. Let C be the (0,3)-tensor defined by

$$C(X,Y,Z) \equiv (\nabla_Z g)(X,Y) = Zg(X,Y) - g(\nabla_Z X,Y) - g(X,\nabla_Z Y). \tag{11}$$

Clearly $C(X,Y,Z) = C(Y,X,Z)$, due to symmetry of g. The tensor C is sometimes referred to as the *cubic form* associated to the pair (∇, g). Clearly g is parallel under ∇ if and only if $C \equiv 0$.

For any connection ∇, its g-conjugate connection ∇^* is defined by

$$Zg(X,Y) = g(\nabla_Z X,Y) + g(X,\nabla^*_Z Y). \tag{12}$$

It can be checked easily that ∇^* is indeed a connection. In addition, g-conjugation of a connection is involutive, i.e., $(\nabla^*)^* = \nabla$.

Comparing (11) and (12), we have

$$C(X, Y, Z) = g(X, (\nabla^* - \nabla)_Z Y). \tag{13}$$

Immediately we see that

$$C^*(X, Y, Z) \equiv (\nabla^*_Z g)(X, Y) = -C(X, Y, Z).$$

Therefore $C(X, Y, Z) = C^*(X, Y, Z) = 0$ if and only if $\nabla^* = \nabla$, that is, ∇ is g-self-conjugate. A connection is both g-self-conjugate and torsion-free must coincide with the Levi-Civita connection ∇^{LC} associated to g (see also Remark 2.11 below).

Definition 2.9. We say the pseudo-Riemannian metric g and the connection ∇ are *Codazzi-coupled* if the following identity holds

$$(\nabla_Z g)(X, Y) = (\nabla_X g)(Z, Y). \tag{14}$$

Stated in terms of the cubic form C, this condition is

$$C(X, Y, Z) = C(Z, Y, X).$$

Because $C(X, Y, Z) = C(Y, X, Z)$, the condition for (g, ∇) being Codazzi-coupled is equivalent to C being totally symmetric in all of its indices.

A simple calculation shows that

$$C(X, Y, Z) - C(X, Z, Y) = g(X, T^{\nabla^*}(Z, Y) - T^{\nabla}(Z, Y)).$$

So C is totally symmetric if and only if ∇ and ∇^* have same torsion. We summarize the above discussions by the following proposition.

Proposition 2.10. *Let g be any pseudo-Riemannian metric and ∇ be any affine connection. Let ∇^* be the g-conjugate connection of ∇, and $C = \nabla g$, $C^* = \nabla^* g$. Then the following statements are equivalent:*

(a). (∇, g) *is Codazzi-coupled;*
(b). (∇^*, g) *is Codazzi-coupled;*
(c). C *is totally symmetric;*
(d). C^* *is totally symmetric;*
(e). $T^{\nabla} = T^{\nabla^*}$.

A manifold M equipped with a pseudo-Riemannian metric g and a torsion-free connection ∇ is called a *statistical manifold* if (g, ∇) is Codazzi-coupled [11]. This is the subject of "classical" information geometry. In "quantum" information geometry, we also allow ∇ to carry torsion. Matsuzoe [15] introduced *Statistical Manifold Admitting Torsion (SMAT)* as a manifold (M, g, ∇) satisfying

$$(\nabla_Y g)(X, Z) - (\nabla_X g)(Y, Z) = g(T^{\nabla}(X, Y), Z).$$

Such a definition ensures that the conjugate connection ∇^* is torsion-free. Note that for a SMAT, ∇ and ∇^* are not necessarily Codazzi coupled with the metric g.

Given a pair of conjugate connections ∇ and ∇^*, one can construct a family of connections $\{\nabla^{(\alpha)}\}_{\alpha\in\mathbb{R}}$, called α-connections [1]

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*, \tag{15}$$

with $\nabla^{(1)} = \nabla$ and $\nabla^{(-1)} = \nabla^*$. It is obvious from this definition that $(\nabla^{(\alpha)})^* = \nabla^{(-\alpha)}$ and the cubic form $C^{(\alpha)}$ for the α-connections is given by

$$C^{(\alpha)}(X, Y, Z) = \alpha C(X, Y, Z),$$

and

$$g(\nabla_Z^{(\alpha)} X, Y) = g(\nabla_Z^{(0)} X, Y) - \frac{\alpha}{2} C(X, Y, Z).$$

Remark 2.11. Note that the 0-connection, which renders the metric g parallel, may admit torsion in general, and hence differ from the Levi-Civita connection ∇^{LC}. To see this, given a metric g, any connection ∇ can be written as a combination of the Levi-Civita connection with a unique (0,3)-tensor $A(X, Y, Z)$ known as the potential of ∇

$$g(\nabla_X Y, Z) = g(\nabla_X^{LC} Y, Z) + A(X, Y, Z).$$

It is not hard to find that C and A are related by

$$C(Y, Z, X) = -A(X, Y, Z) - A(X, Z, Y).$$

All the constraints on ∇ can be expressed in terms of A. As an example, the torsion-freeness plus Codazzi coupling with g is equivalent to that A is totally symmetric; in this case we have $C = -2A$.

2.4. Coupling of ∇ with ω

So far we have discussed Codazzi coupling of a torsion-free connection ∇ with a quadratic operator L and with a pseudo-Riemannian metric g, as generalization of parallelism. We now ask whether we can extend such consideration to the 2-form $\omega(X, Y) = g(LX, Y)$, and have a notion of Codazzi coupling of ∇ with ω that extends $\nabla\omega = 0$. The answer is negative, as we now show.

Let us define, in analogous to the cubic form C, the following (0,3)-tensor

$$\Gamma(X, Y, Z) = (\nabla_Z \omega)(X, Y) = Z\omega(X, Y) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y),$$

which is skew-symmetric in X, Y: $\Gamma(X, Y, Z) = -\Gamma(Y, X, Z)$. However, imposing naively the Codazzi coupling condition

$$(\nabla_Z \omega)(X, Y) = (\nabla_X \omega)(Z, Y)$$

would require $\Gamma(X, Y, Z)$ to be symmetric in X and Z. Therefore

$$\begin{aligned}\Gamma(X, Y, Z) &= \Gamma(Z, Y, X) = -\Gamma(Y, Z, X) = -\Gamma(X, Z, Y) = \Gamma(Z, X, Y)\\ &= \Gamma(Y, X, Z) = -\Gamma(X, Y, Z).\end{aligned}$$

Hence $\Gamma(X, Y, Z) = 0$, that is, $\nabla\omega = 0$. We conclude that naive Codazzi coupling of ∇ with ω implies that ω is parallel under ∇. When $\nabla\omega = 0$ and ∇ is torsion-free, one deduce that ω is automatically closed (see Lemma 3.1

below) and ∇ is said to be a *symplectic connection*. A symplectic manifold (M, ω) equipped with a symplectic connection is known as a *Fedosov manifold* [8].

On the other hand, in analogous to g-conjugation of a connection, we may define, for any connection ∇, the ω-conjugate connection $\nabla^\dagger$ by setting

$$Z\omega(X, Y) = \omega(\nabla_Z X, Y) + \omega(X, \nabla^\dagger_Z Y). \tag{16}$$

It can be verified that $\nabla^\dagger$ is indeed a connection, and that the $\dagger$ operation on ∇ is involutive: $(\nabla^\dagger)^\dagger = \nabla$. Despite of the skew-symmetric nature of ω, ω-conjugation is one and the same whether defined with respect to the first or second slot of ω. This is to say, (16) holds if and only if

$$Z\omega(X, Y) = \omega(\nabla^\dagger_Z X, Y) + \omega(X, \nabla_Z Y).$$

Making use of the definition of Γ, we have

$$(\nabla^\dagger_Z \omega)(X, Y) = -\Gamma(X, Y, Z) = -(\nabla_Z \omega)(X, Y).$$

It follows that $T^\nabla = T^{\nabla^\dagger}$ if and only if $\nabla = \nabla^\dagger$. This is very different from the case of ∇ and its g-conjugate ∇^*. We summarize the above discussions by the following proposition.

Proposition 2.12. *Let ω be any skew-symmetric 2-form, ∇ be an arbitrary affine connection, and $\nabla^\dagger$ be the ω-conjugate connection of ∇. Then the following statements are equivalent:*

(a). $\nabla\omega = 0$;
(b). $\nabla = \nabla^\dagger$;
(c). $T^\nabla = T^{\nabla^\dagger}$.

2.5. Klein Group of Transformations of ∇

We now show a key relationship between the three transformations of a connection ∇: its g-conjugate ∇^*, its ω-conjugate $\nabla^\dagger$, and its L-gauge transform ∇^L.

Theorem 2.13. *Let (g, ω, L) be a compatible triple, and ∇^*, $\nabla^\dagger$, and ∇^L denote, respectively, g-conjugation, ω-conjugation, and L-gauge transformation of an arbitrary connection ∇. Then, $(id, *, \dagger, L)$ realizes a 4-element Klein group action on the space of affine connections:*

$$\begin{aligned}
&(\nabla^*)^* = (\nabla^\dagger)^\dagger = (\nabla^L)^L = \nabla;\\
&\nabla^* = (\nabla^\dagger)^L = (\nabla^L)^\dagger;\\
&\nabla^\dagger = (\nabla^*)^L = (\nabla^L)^*;\\
&\nabla^L = (\nabla^*)^\dagger = (\nabla^\dagger)^*.
\end{aligned}$$

Proof. The definition of conjugate connection with respect to g and to ω leads to $(\nabla^*)^* = \nabla, (\nabla^\dagger)^\dagger = \nabla$, while Corollary 2.3 shows that $(\nabla^L)^L = \nabla$. Next,

$$\begin{aligned}\omega(\nabla^\dagger_Z X, Y) &= Z(\omega(X,Y)) - \omega(X, \nabla_Z Y)\\ &= Z(g(LX,Y)) - g(LX, \nabla_Z Y)\\ &= g(\nabla^*_Z(LX), Y)\\ &= g(L((\nabla^*)^L_Z X), Y)\\ &= \omega((\nabla^*)^L_Z X, Y),\end{aligned}$$

which establishes $\nabla^\dagger = (\nabla^*)^L$. Hence, upon applying L-gauge transformation to both sides, $(\nabla^\dagger)^L = \nabla^*$; upon substituting ∇^* for ∇ on both sides, $(\nabla^*)^\dagger = \nabla^L$. Commutativity of each of the three pairs of transformations is established by the fact that each transformation, being involutive, equals their inverse. □

Theorem 2.13, along with Proposition 2.12, immediately leads to

Corollary 2.14. *Given a compatible triple* (g, ω, L)*,* $\nabla\omega = 0$ *if and only if*

$$\nabla^* = \nabla^L.$$

Explicitly written,

$$\nabla^*_Z X = \nabla_Z X + L^{-1}((\nabla_Z L)X) = \nabla_Z X + L((\nabla_Z L^{-1})X). \tag{17}$$

Remark 2.15. Note that, in both Theorem 2.13 and Corollary 2.14, there is no requirement of ∇ to be torsion-free nor is there any assumption about its Codazzi coupling with L or with g. In particular, Corollary 2.14 says that, when viewing $\omega(X,Y) = g(LX,Y)$, $\nabla\omega = 0$ if and only if the torsions introduced by $*$ and by L are cancelled.

There have been confusing statements about (17), even for the special case of $L = J$, the almost complex structure. In reference [6, Proposition 2.5(2)], (17) was shown after assuming (g, ∇) to be a statistical structure. On the other hand, [17, Lemma 4.2] claimed the converse, also under the assumption of (M, g, ∇) being statistical. As Corollary 2.14 shows, the Codazzi coupling of ∇ and g is not relevant for (17) to hold; (17) is entirely a consequence of $\nabla\omega = 0$. Corollary 2.14 is a special case of a more general theorem ([23], Theorem 21).

3. Codazzi-(para-)Kähler Structure

In this section, we restrict ourselves to the case that L is an almost complex structure J or an almost para-complex structure K. We assume that L is compatible with g and ω in the sense of (3) and (5), so (g, ω, L) forms a compatible triple. We study the interactions between these three objects with the same torsion-free connection ∇: Codazzi coupling of ∇ with L, Codazzi coupling of ∇ with g, and $\nabla\omega = 0$. We show that Codazzi couplings of a torsion-free ∇ with both g and L, or Codazzi coupling of an almost-symplectic ∇ with either

g or L, leads to (para-)Kähler structure. This allows us to define the notion of Codazzi-(para-)Kähler manifolds as simultaneously statistical manifolds and (para-)Kähler manifolds, which include special (para-)Kähler geometry as a special case. In particular, we obtain a characterization of Codazzi-(para-)Kähler structure as a quadruple (g, ω, L, ∇) for which the torsion-freeness of ∇ is preserved under any two of the following three operations: g-conjugation, ω-conjugation, or L-gauge transformation.

3.1. Simultaneous Codazzi Couplings by the Same ∇

We start with the following simple lemma.

Lemma 3.1. *If ∇ is torsion-free, then we have*

$$\mathrm{d}\omega(X,Y,Z) = (\nabla_Z\omega)(X,Y) + (\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X). \tag{18}$$

Proof. By the Cartan's formula, we have

$$\begin{aligned}\mathrm{d}\omega(X,Y,Z) = &\, X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y)\\ &- \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y).\end{aligned}$$

Since ∇ is torsion-free,

$$\begin{aligned}\mathrm{d}\omega(X,Y,Z) =& X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) - \omega(\nabla_X Y - \nabla_Y X, Z)\\ &- \omega(\nabla_Y Z - \nabla_Z Y, X) - \omega(\nabla_Z X - \nabla_X Z, Y)\\ =& (X\omega(Y,Z) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z))\\ &+ (Y\omega(Z,X) - \omega(\nabla_Y Z, X) - \omega(Z, \nabla_Y X))\\ &+ (Z\omega(X,Y) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y))\\ =& (\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla_Z\omega)(X,Y).\end{aligned}$$

□

We are now ready to introduce our main Theorem.

Theorem 3.2. *Let ∇ be a torsion-free connection and g a pseudo-Riemannian metric on M. Let L denote either J or K that is compatible with g. Assuming*

(i) (∇, g) *is Codazzi-coupled;*
(ii) (∇, L) *is Codazzi-coupled.*

Then, (M, g, L) is a (para-)Kähler manifold.

Proof. To show M is a (para-)Kähler manifold, we need to prove both L is integrable and ω is closed. The former follows immediately from Proposition 2.6 and the Condition (ii) above. So we only needs to show $\mathrm{d}\omega = 0$.

We first derive the following identity:

$$\begin{aligned}(\nabla_Z\omega)(X,Y) &= Z(\omega(X,Y)) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y)\\ &= Z(g(LX,Y)) - g(L\nabla_Z X, Y) - g(LX, \nabla_Z Y)\\ &= (\nabla_Z g)(LX,Y) + g((\nabla_Z L)X, Y)\end{aligned}$$

$$= C(LX,Y,Z) + g((\nabla_Z L)X,Y). \tag{19}$$

Alternate X,Y,Z in (19) and sum them up, making use of (18), we find

$$\begin{aligned} &\mathrm{d}\omega(X,Y,Z) - C(LX,Y,Z) - C(LY,Z,X) - C(LZ,X,Y) \\ &= \mathrm{d}\omega(Z,Y,X) - C(LZ,Y,X) - C(LX,Z,Y) - C(LY,X,Z). \end{aligned}$$

By Proposition 2.10, C is totally symmetric; this shows $\mathrm{d}\omega(X,Y,Z) = \mathrm{d}\omega(Z,Y,X)$. On the other hand, $\mathrm{d}\omega$ is totally skew-symmetric as well, so we conclude $\mathrm{d}\omega = 0$. □

Remark 3.3. Both Conditions (i) and (ii) are needed for $\mathrm{d}\omega = 0$. If only Condition (i) holds, we may take (M,g,ω,∇) be any non-(para-)Kähler manifold with its Levi-Civita connection. We see immediately that $\nabla\omega \neq 0$.

On the other hand, if only Condition (ii) holds, we only have

$$\begin{aligned} 2\mathrm{d}\omega(X,Y,Z) = {} & C(LX,Y,Z) - C(LX,Z,Y) + C(LY,Z,X) - C(LY,X,Z) \\ & + C(LZ,X,Y) - C(LZ,Y,X). \end{aligned}$$

Theorem 3.4. *Let ∇ be a torsion-free connection on M, and L denote either J or K, so $L^2 = \mp id$. Then, for the following three statements regarding any compatible triple (g,ω,L), any two imply the third:*

(a). (∇, g) *is Codazzi-coupled;*
(b). (∇, L) *is Codazzi-coupled;*
(c). $\nabla\omega = 0$.

As a result, M is a Kähler or para-Kähler manifold equipped with a symplectic connection ∇.

Proof. For convenience, let us denote $B(X,Y,Z) = g((\nabla_X L)Y,Z)$. From (10) we can easily deduce that

$$B(X,LY,Z) = B(X,Y,LZ). \tag{20}$$

Because of (19), $\nabla\omega$ can be expressed as

$$(\nabla_X\omega)(Y,Z) = B(X,Y,Z) + C(LY,Z,X), \tag{21}$$

so its skew-symmetry $(\nabla_X\omega)(Y,Z) = -(\nabla_X\omega)(Z,Y)$ gives rise to the identity

$$B(X,Y,Z) + C(LY,Z,X) + B(X,Z,Y) + C(LZ,Y,X) = 0. \tag{22}$$

Statement (c) $\nabla\omega = 0$ translates to

$$B(X,Y,Z) + C(LY,Z,X) = 0. \tag{23}$$

Statement (b), namely Codazzi coupling of ∇ and L, translates to

$$B(X,Y,Z) = B(Y,X,Z). \tag{24}$$

From (a) and (b) to (c): From (18) and using (21),

$$\begin{aligned}\mathrm{d}\omega(X,Y,Z) = B(X,Y,Z) + C(LY,Z,X) + B(Y,Z,X) + C(LZ,X,Y)\\ + B(Z,X,Y) + C(LX,Y,Z).\end{aligned}$$

Applying (22) to the above, along with (24) and total symmetry of C, we have

$$0 = \mathrm{d}\omega(X,Y,Z) = B(Y,Z,X) + C(LX,Y,Z),$$

where Theorem 3.2 is invoked in the first equality. Therefore, upon substituting LX in place of X and using (20), we get

$$\begin{aligned}0 &= B(Y,Z,LX) + C(L^2X,Y,Z)\\ &= B(Y,LZ,X) + C(L^2Z,X,Y)\\ &= (\nabla_Y\omega)(LZ,X),\end{aligned}$$

where the last step is from (21). As X, Y, Z are all arbitrary vector fields, we conclude that $\nabla\omega = 0$.

From (a) and (c) to (b): We start from (23) which is the expression of Statement (c). Writing out (20) in terms of C, we get

$$C(L^2Y,X,Z) = C(LY,LZ,X).$$

Total symmetry of C as guaranteed by Statement (a) allows us to have

$$C(LY,LZ,X) = C(LX,LZ,Y),$$

which, in terms of B, is simply

$$B(X,Y,LZ) = B(Y,X,LZ).$$

So symmetry of B in the first two slots is proven, which is Statement (b).

From (b) and (c) to (a): Using identity (20), along with (24) as guaranteed by Statement (b), allows

$$B(X,LY,Z) = B(X,Y,LZ) = B(Y,X,LZ) = B(Y,LX,Z).$$

Invoking (23) as guaranteed by Statement (c), we have

$$C(L^2Y,Z,X) = C(L^2X,Z,Y),$$

which leads to $C(Y,Z,X) = C(X,Z,Y)$, hence proving Statement (a). □

Remark 3.5. Theorem 3.2 says that, for an arbitrary statistical manifold (M, g, ∇), if there exists a (para-)complex structure $J(K)$ compatible with g such that ∇ and $J(K)$ are Codazzi-coupled, then what we have of (M, g, ∇, L) is a (para-)Kähler manifold. Theorem 3.4 further says that ∇ is then a symplectic connection, i.e. $\nabla\omega = 0$, and (M, ω, ∇) is a Fedosov manifold.

Theorem 3.4 also says that, for any Fedosov manifold (M, ω, ∇), if there exists a (para-)complex structure $J(K)$ compatible with ω such that ∇ and $J(K)$ are Codazzi-coupled, then (M, ω, ∇, L) is a (para-)Kähler manifold. In other words, Codazzi coupling of ∇ with L turns a statistical manifold *or* a

Fedosov manifold into a (para-)Kähler manifold, which is then both statistical and symplectic.

Corollary 3.6. *Given compatible triple (g, ω, L) on a manifold M. Then any two of the following three statements imply the third, meanwhile making M a (para-)Kähler manifold:*

(a). (M, g, ∇) *is a statistical manifold;*
(b). (M, ω, ∇) *is a Fedosov manifold;*
(c). (∇, L) *is Codazzi coupled.*

We remark that Theorems 3.2 and 3.4 (and Corollary 3.6) can also be derived when we consider the conditions of torsion-preservation by the g-conjugate ∇^*, ω-conjugate $\nabla^\dagger$, and L-gauge ∇^L transformations of a torsion-free connection ∇. Based on Propositions 2.2, 2.10, and 2.12, we can restart the above results as

Theorem 3.7. *Let ∇ be a torsion-free connection on M, and ∇^*, $\nabla^\dagger$, and ∇^L are the transforms induced by the compatible triple (g, ω, L). Then, M is (para-)Kähler if any two of the following three statements are true:*

(a). ∇^* *is torsion-free;*
(b). $\nabla^\dagger$ *is torsion-free;*
(c). ∇^L *is torsion-free.*

3.2. Codazzi-(para-)Kähler Structures

Throughout our previous discussions of the (para-)Kähler structures, ∇ is not necessarily the Levi-Civita connection associated to g. So it is very natural to propose the following definitions:

Definition 3.8. An *almost Codazzi-(para-)Kähler manifold* $(M, g, J(K), \nabla)$ is by definition an almost (para-)Hermitian manifold $(M, g, J(K))$ with an affine connection ∇ (not necessarily torsion-free) which is Codazzi-coupled to both g and $J(K)$. If ∇ is torsion-free, then $J(K)$ is automatically integrable and ω is parallel, so in this case we will call $(M, g, J(K), \nabla)$ a *Codazzi-(para-)Kähler manifold* instead.

In practice, one probably should think of an almost Codazzi-(para-) Kähler manifold as an almost (para-)Hermitian manifold with a specified nice affine connection. Such structure exists on all almost (para-)Hermitian manifolds (M, g, L). In particular, one can take ∇ to be any (para-)Hermitian connection [7]([10]), which satisfies

$$\nabla g = 0 \quad \text{and} \quad \nabla L = 0.$$

In the like manner, any (para-)Kähler manifold with its Levi-Civita connection is automatically Codazzi-(para-)Kähler.

Recall that "statistical structure" (M, g, ∇) is characterized by Condition (i) of Theorem (3.7), see Sect. 3.1. Hence, we can view Theorem (3.7) as a characterzation theorem for Codazzi-(para-)Kähler structure. So a statistical structure can be "enhanced" to a Codazzi-(para-)Kähler structure (M, g, L, ∇) by supplying it with an L that is compatible with g and that is Codazzi coupled with ∇. Hence, Codazzi-(para-)Kähler structure is a (para-)Kähler structure which is at the same time statistical. A statistical manifold (M, g, ∇) can also be "enhanced" to a Codazzi-(para-)Kähler structure (M, g, L, ∇) by supplying it with an ω such that ∇ is symplectic. Hence, Codazzi-(para-)Kähler manifold is a Fedosov manifold which is at the same time a statistical manifold.

Insofar as Codazzi-(para-)Kähler manifolds, being a special kind of (para-)Kähler manifold, generalize the way ∇ is integrated into the compatible triple (g, ω, L), we propose the notion of "compatible quadruple" to describe the compatibility between the four objects g, ω, L, and ∇ on a manifold.

Definition 3.9. A *compatible quadruple* on a manifold M is a quadruple (g, ω, L, ∇), where g and ω are symmetric and skew-symmetric non-degenerate two-forms respectively, L is either an almost complex or almost para-complex structure, and ∇ is a torsion-free connection, that satisfy the following relations:

(i) $\omega(X, Y) = g(LX, Y)$;
(ii) $g(LX, Y) + g(X, LY) = 0$;
(iii) $\omega(LX, Y) = \omega(LY, X)$;
(iv) $(\nabla_X L)Y = (\nabla_Y L)X$;
(v) $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$;
(vi) $(\nabla_X \omega)(Y, Z) = 0$.

for any vector fields X, Y, Z on M.

Therefore, restating Theorem 3.4 and Lemma 3.6 in terms of any three of the compatible quadruple, we have

Proposition 3.10. *A smooth manifold M is Codazzi-(para-)Kähler if and only if any of the following conditions holds:*

(a). (g, L, ∇) *satisfy (ii), (iv) and (v);*
(b). (ω, L, ∇) *satisfy (iii), (iv) and (vi);*
(c). (g, ω, ∇) *satisfy (v) and (vi), in which case L is determined by (i).*

By Theorem 2.13, any Codazzi-(para-)Kähler manifold admits a dual pair (∇, ∇^C) of torsion-free connections, where ∇^C is called the *Codazzi dual* of ∇:

$$\nabla^C = \nabla^* = \nabla^L.$$

Proposition 3.11. *For any Codazzi-(para-)Kähler manifold, its Codazzi dual connection satisfies:*

(i) $(\nabla^C_X L)Y = (\nabla^C_Y L)X$;
(ii) $(\nabla^C_X g)(Y, Z) = (\nabla^C_Y g)(X, Z)$;
(iii) $(\nabla^C_X \omega)(Y, Z) = 0$.

A very important class of nontrivial Codazzi-Kähler manifolds is the so-called special Kähler manifold, in which ∇ is flat (and hence ∇^C is also flat). In special Kähler geometry, the holomorphic cubic form Ξ (also known as the Yukawa coupling in mirror symmetry) is essentially defined to be

$$\Xi(X, Y, Z) = -\omega(JX, (\nabla_Z J)Y).$$

From this definition, we see immediately that

$$\begin{aligned}
\Xi(X, Y, Z) &= g(X, \nabla_Z JY) - g(X, J\nabla_Z Y) \\
&= Zg(X, JY) - (\nabla_Z g)(X, JY) - g(\nabla_Z X, JY) - g(X, J\nabla_Z Y) \\
&= Z\omega(Y, X) - \omega(\nabla_Z Y, X) - \omega(Y, \nabla_Z X) - C(X, JY, Z) \\
&= (\nabla_Z \omega)(Y, X) - C(X, JY, Z) \\
&= -C(X, JY, Z).
\end{aligned}$$

From Proposition 2.10 we know that Ξ is totally symmetric. And we can use the above expression to define a symmetric cubic form on any Codazzi-(para-)Kähler manifolds.

4. Summary and Discussions

It is a well-known fact that a (para-)Kähler manifold (M, g, L) (with $L = J, K$) is characterized by the existence of a torsion-free connection ∇ that renders both g and L parallel. In this paper, we relaxed the parallelism condition to the Codazzi couplings of ∇ with g and L. We first showed in Proposition 2.6 that for any torsion-free connection ∇, its Codazzi coupling with a quadratic operator L (for which J and K are special cases) leads to the integrability of L, and hence transversal foliations on M. Then we showed, in Theorem 3.2, Theorem 3.4 and Corollary 3.6, that Codazzi coupling of ∇ with any two of the compatible triple (g, ω, L) implies its coupling with the third, giving rise to Codazzi-(para-)Kähler structure on the manifold. In particular, a manifold (M, g, ω, ∇) is Codazzi-(para-)Kähler if (M, ω, ∇) is Fedosov *and* (M, g, ∇) is statistical, provided that there exists an almost (para-)complex operator L on M compatible with both g and with ω. This points at complementary status of symplectic structure and statistical structure in making up a (para-)Kähler structure. In particular, g-conjugation ∇^* of a connection is exactly the L-gauge transformation ∇^L for $L = J, K$ when ∇ is symplectic with respect to the compatible ω. Our investigations about Codazzi coupling of ∇ illuminate how a torsion-free connection ∇ may fit snugly within the compatible triple on a (para-)Kähler manifold M, such that M accommodates a "compatible quadruple" (g, ω, L, ∇).

Codazzi coupling is the cornerstone of affine differential geometry (e.g., [12–14,19]), and in particular so for information geometry. In information geometry, the Riemannian metric g and a pair of torsion-free g-conjugate affine connections ∇, ∇^* are naturally induced by the so-called divergence (or "contrast") function on a manifold M (see [1]). While a statistical structure is naturally induced on M, the divergence function will additionally induce a symplectic structure ω on the product manifold $M \times M$, see [2,27]. Furthermore, Zhang and Li [27] imposed conditions on divergence functions to make the induced symplectic structure compatible with an almost complex structure J in order to obtain a Kähler structure. Our results here provide a complete answer to the question of precise conditions under which a statistical manifold could be "enhanced" to a Kähler and/or para-Kähler manifold, and clarify some confusions in the literature regarding the roles of Codazzi coupling of ∇ with g and with L in the interactions between statistical structure (as generalized Riemannian structure), symplectic structure, and (para-)complex structure.

Codazzi-(para-)Kähler manifolds are generalizations of special Kähler manifolds by removing the requirement of ∇ to be (dually) flat in the latter. Special Kähler manifolds are first mathematically formulated by Freed [5], and they have been extensively studied in physics literature since 1980's. For example, special Kähler structures are found on the base of algebraic integrable systems [4] and moduli space of complex Lagrangian submanifolds in a hyperkähler manifold [9]. From the above discussions, we can view special Kähler manifolds as "enhanced" from the class of dually-flat statistical manifold, namely, Hessian manifolds [20]. In information geometry, non-flat affine connections are abundant—the family of $\nabla^{(\alpha)}$ connections (15) associated with a pair of dually-flat connections ∇, ∇^* are non-flat except $\alpha = \pm 1$ [25], forming α-Hessian manifold [28]. Our analysis (in Sect. 2) clearly shows that it is the preservation of torsion under the conjugate/gauge transformations of ∇ (by g, ω, L) that highlights the essence of Codazzi coupling (of each) with such ∇. So our generalization of special Kähler geometry to Codazzi-Kähler geometry, which shifts attention from curvature to torsion, may be meaningful for the investigation of bidualistic geometric structures in statistical and information sciences [24,26].

Acknowledgements

This collaborative research started while the first author (F.T.) was a Ph.D. student at MIT and the second author (J.Z.) was on sabbatical visit at the Center for Mathematical Sciences and Applications at Harvard University in the Fall of 2014 under the auspices of Prof. S.-T. Yau. The first author (T.F.) would also like to express his gratitude towards Prof. Yau's constant encouragement and help. Dissemination of this research is supported by ARO/DARPA Grant W11NF-16-1-0383 (PI: Jun Zhang).

References

[1] Amari, S., Nagaoka, H.: Methods of Information Geometry, Volume 191 of Translations of Mathematical Monographs. AMS, Providence (2000)

[2] Barndorff-Nielsen, O.E., Jupp, P.E.: Yokes and symplectic structures. J. Stat. Plan. Inference **63**(2), 133–146 (1997)

[3] Cheng, S.-Y., Yau, S.-T.: The real Monge-Ampère equation and affine flat structures. In: Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. I, pp. 339–370. Science Press, New York (1982)

[4] Donagi, R., Witten, E.: Supersymmetric Yang-Mills theory and integrable systems. Nucl. Phys. B **460**(2), 299–334 (1996)

[5] Freed, D.S.: Special Kähler manifolds. Commun. Math. Phys. **203**(1), 31–52 (1999)

[6] Furuhata, H.: Hypersurfaces in statistical manifolds. Differ. Geom. Appl. **27**(3), 420–429 (2009)

[7] Gauduchon, P.: Hermitian connections and Dirac operators. Bollettino della Unione Matematica Italiana-B **11**(2, Suppl.), 257–288 (1997)

[8] Gelfand, I., Retakh, V., Shubin, M.: Fedosov manifolds. Adv. Math. **136**(1), 104–140 (1998)

[9] Hitchin, N.J.: The moduli space of complex Lagrangian submanifolds. In: Surveys in Differential Geometry, vol. VII, pp. 327–345. International Press (2000)

[10] Ivanov, S., Zamkovoy, S.: Parahermitian and paraquaternionic manifolds. Differ. Geom. Appl. **23**(2), 205–234 (2005)

[11] Lauritzen, S.L.: Statistical manifolds. Differential Geometry in Statistical Inference. Volume 10 of IMS Lecture Notes Monograph Series, pp. 163–216. Institute of Mathematical Statistics, The Hague (1987)

[12] Leder, J., Schwenk-Schellschmidt, A., Simon, U., Wiehe, M.: Generating higher order Codazzi tensors by functions. In: Geometry And Topology Of Submanifolds IX, pp. 174–191 (1999)

[13] Liu, H.L., Simon, U., Wang, C.P.: Codazzi tensors and the topology of surfaces. Ann. Glob. Anal. Geom. **16**(2), 189–202 (1998)

[14] Liu, H.L., Simon, U., Wang, C.P.: Higher order Codazzi tensors on conformally flat spaces. Contrib. Algebra Geom. **39**(2), 329–348 (1998)

[15] Matsuzoe, H.: Statistical manifolds and geometry of estimating functions. Prospects of Differential Geometry and Its Related Fields. Proceedings of the 3rd International Colloquium on Differential Geometry and Its Related Fields, pp. 187–202. World Scientific, Singapore (2013)

[16] Moroianu, A.: Lectures on Kähler Geometry, Volume 69 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge (2007)

[17] Noda, T.: Symplectic structures on statistical manifolds. J. Aust. Math. Soc. **90**(3), 371–384 (2011)

[18] Nomizu, K., Sasaki, T.: Affine Differential Geometry: Geometry of Affine Immersions, Volume 111 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1994)

[19] Pinkall, U., Schwenk-Schellschmidt, A., Simon, U.: Geometric methods for solving Codazzi and Monge–Ampere equations. Math. Ann. **298**(1), 89–100 (1994)

[20] Shima, H.: On certain locally flat homogeneous manifolds of solvable Lie groups. Osaka J. Math. **13**(2), 213–229 (1976)

[21] Simon, U.: Affine differential geometry. In: Dillen, F., Verstraelen, L. (eds.) Handbook of Differential Geometry, vol. 1, pp. 905–961. North-Holland (2000)

[22] Schwenk-Schellschmidt, A., Simon, U.: Codazzi-equivalent affine connections. Result Math. **56**(1–4), 211–229 (2009)

[23] Tao, J., Zhang, J.: Transformations and coupling relations for affine connections. Differ. Geom. Appl. **49**, 111–130 (2016)

[24] Zhang, J.: Divergence function, duality, and convex analysis. Neural Comput. **16**(1), 159–195 (2004)

[25] Zhang, J.: A note on curvature of α-connections of a statistical manifold. Ann. Inst. Stat. Math. **59**(1), 161–170 (2007)

[26] Zhang, J.: Nonparametric information geometry: from divergence function to referential-representational biduality on statistical manifolds. Entropy **15**(12), 5384–5418 (2013)

[27] Zhang, J., Li, F.-B.: Symplectic and Kähler structures on statistical manifolds induced from divergence functions. In: Geometric Science of Information, volume 8085 of Lecture Notes in Computer Science, pp. 595–603. Springer, New York (2013)

[28] Zhang, J., Matsuzoe, H.: Dualistic differential geometry associated with a convex function. In: Advances in Applied Mathematics and Global Optimization: In: Honor of Gilbert Strang, Volume 17 of Advances in Mechanics and Mathematics, pp. 439–464 (2009)

Teng Fei
Columbia University
New York NY
USA
e-mail: tfei@math.columbia.edu

Jun Zhang
University of Michigan
Ann Arbor MI
USA
e-mail: junz@umich.edu

Received: March 16, 2017.
Accepted: June 14, 2017.