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4 Subset System Mathematical Abstraction of Object and Context

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4.1 INTRODUCTION

Set theory is a universal language for modeling mathematical objects. Modern set theory, initiated by Cantor and Dedekind in the 1870s and amended by Zermelo-Fraenkel axioms, is held by mainstream mathematicians as providing a solid foundation for all branches of mathematics, and has enjoyed great success in providing a universal language to construct necessary tools for handling physical and engineering systems. To explore the potential power of mathematics as applied to social, behavioral, cognitive, and information systems, there is a need to identify set-theoretic primitives for modeling relational structures that are prevalent in these systems yet distinct from physical applications. In this chapter, we offer an initial attempt at mathematical primitives for modeling relational structures based on naïve set theory. Our theory is based on a mathematical concept that we call “subset system” here, which is naturally equipped with *two* pairs of relations, namely “pre-order” and “tolerance,” that would allow us to model, in a most abstract fashion, objects, features, contexts, etc.

A *subset system* is a pair of sets (V, E) , where V is a set, called ground-set, and E is a collection of subsets on V . So a subset system is a set (the V -part of the definition) along with a family of its subsets (the E -part of the definition). Denote $\mathcal{P}(V) \equiv 2^V$ as the power-set of V . E itself can be viewed as a subset $E \subset \mathcal{P}(V)$ (and hence a point of $\mathcal{P}\mathcal{P}(V)$, the power of the power-set). To alleviate confusion, we will use the word “*element*” $v \in V$ when referring to the ground-set V , and the word “*member*” $e \in E$ when referring to the subset collection E .

In the graph-theoretic literature, a subset system (V, E) is also called a “hypergraph,” where each element $v \in V$ is called a vertex, and each member e (which is a subset of V) of E is called a “hyperedge” (i.e., an “edge” connecting multiple vertices) or, with an abuse of terminology, “face.” We used the term subset system in this [paper](#), because of our much broader perspective than the graph-theoretic approach. A subset system can also be viewed as defining a binary relation R between a set V and another set E such that $\nu R e$ if and only if $\nu \in e$. Under this

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viewpoint, a subset system is a special case of the so-called “cross-table” $V \times E$, the data structure which lists the co-occurrences of elements of a pair of sets (here E is treated, *a priori*, as a set in the same way as V is), as used in the so-called *Formal Concept Analysis* (Ganter & Wille, 2012). As a concrete example, think about the coauthorship network in the study of patterns of scientific collaborations. Here, V is a list of authors, and E is the list of papers some of these authors have collaboratively written—single-authored papers are included, but multiple publications by the same collaborators are condensed and counted as single occurrence.

A subset system is an abstraction of many algebraic and topological structures that lie at the core of modern mathematics. The notion is quite generic, as it does not impose, *a priori*, any requirement about the inclusion/exclusion of particular members in the collection of subsets that make up the subset system. Examples of subset systems abound as familiar mathematical objects:

- *Topped \cap system*: The subset system is required to be closed with respect to the set-intersection operation and must also contain the full set V . In this case, the system becomes a complete lattice ordered by the set-inclusion relation.
- *Topology*: The subset system is required to be closed with respect to arbitrary (countably many) unions and finite intersections. In this case, the system becomes a bounded distributive lattice. To “topologize” a subset system, that is, using the subset system as a (topological) sub-base to make it a topology, amounts to adding more members to the collection so that the collection is closed under the two operations mentioned above.
- *Alexandrov topology*: The system of subsets needs to satisfy a stronger condition than that of a topology, i.e., arbitrary intersection needs to be closed within the system. In this case, the system becomes a complete distributive lattice. To “Alexandrovize” a topology, that is, making a topology an Alexandrov topology, will require more members to be added to the collection (and hence the collection is larger).
- *Lattice of sets*: The subset system is such that for any two members, there exists a member that includes both and also a member that is included by both. The system forms a lattice ordered by inclusion of subsets.
- *Ring of sets*: The subset system is required to be closed under union and under intersection (or set-theoretic difference). The system forms a distributive lattice.
- *Field of sets*: The subset system, in addition to being a ring of sets, needs also to satisfy closure with respect to the complementation operation. The system forms a Boolean lattice.

- *Borel sets with σ -algebra*: The subset system is required to be closed under complementation and under countably many union and intersection. They form the starting point of modern measure theory.

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From its definition, a subset system has two natural binary relations imposed upon members of E —two members e_1, e_2 of E (which are subsets of V) are said to have a:

- pre-order \preceq relation, denoted as $e_1 \preceq e_2$, iff $e_1 \subset e_2$ as subsets of V ;
- tolerance \simeq relation, denoted as $e_1 \simeq e_2$, iff $e_1 \cap e_2 \neq \emptyset$ as subsets of V .

These two binary relations \preceq and \simeq for members of the collection E are inherited from the set-inclusion relation and non-empty set-intersection relationships among subsets of V . That is, both \preceq and \simeq characterize relations among members of the power-set $\mathcal{P}(V)$. What we are going to study are two *new* relations among elements of the ground-set V . We will show that, by virtue of the *make-up* of E , the particular *collection* of subsets, there can be a pre-order relation (denoted P below) and a tolerance relation (denoted T below) induced on V . This makes elements of the ground-set V in various relations with one another. This is the main contribution of our present work. Our chapter will show that defining these binary relations is “natural,” if we want the collection of subsets to provide a “context” for elements of the ground-set. Our theoretical framework (which is still in its infancy) aims to provide a universal modeling language drawn from order theory (including lattice), topology, measure, and other well-studied algebraic structures, but applicable in weaker contexts than each of the traditional tools.

The relaxation offered by our theory, that is, the consideration of a subset system without any *a priori* assumptions on the collection of subsets, has practical convenience. In social-cognitive-behavioral settings, when objects are modeled as elements of a ground-set, the co-appearance (co-occurrence) of objects can be modeled as a subset. A collection of subsets, therefore, provides contextual information about objects and their relations. We show below that, without making any assumptions about this collection, we can uncover two types of information about the collection of objects by simply using input data in this format. The first is a pre-order relation, which gives information about whether one object is more generic/special than another object. Here, generic (special) is in the sense of appearing in more (less) occasions in the input data. The second is a tolerance relation, which gives information about whether two objects are similar (or its opposite, dissimilar) or not. pre-order and tolerance are two important relations. Mathematically, they provide two ways of falling short of an equivalence class, by violating either symmetry or transitivity. Our framework thus proposes an algorithmic analysis of input data that mimics two rational and complementary styles of cognitive processing—similarity judgment in perception

(enforcing symmetry) and preference judgment in decision (enforcing transitivity).

The plan of this chapter is as follows. Section 2 reviews the mathematical background of binary relations, in particular, pre-order and tolerance relations. In particular, the close relationship between Alexandrov topology and pre-order is reviewed. Section 3 investigates the pre-order and tolerance structures induced on the ground-set of an arbitrary subset system. Independence and information completeness of the two induced relations are studied. Two transformations of subset systems are discussed: complementation and closure with respect to the induced tolerance. A familiar example, biorder, is revisited as a special kind of subset system, one that is “chained.” Finally, Section 4 discusses various ways to modify the structure of a subset system, bringing in the possibility of learning and dynamics into our theoretical framework.

4.2 MATHEMATICAL PRELIMINARIES

4.2.1 Binary Relations

Recall the notions of binary relation, pre-order, and equivalence. The basic description of a binary relation is a collection of ordered binary pairs. Specifically, a binary relation (X, R) is a collection \mathcal{C} of subsets of $X \times X$. Two elements $x, y \in X$ are said to be in relation R , denoted as xRy , when $(x, y) \in \mathcal{C}$.

There are three basic essential aspects of a binary relation:

1. *reflexivity*: for all x , xRx holds;
2. *symmetry*: for all x, y , xRy implies yRx ;
3. *transitivity*: for any $x, y, z \in X$, xRy and yRz implies that xRz .

A binary relation (X, R) is called an

1. *equivalence*: if R is reflexive, symmetric, and transitive.
2. *pre-order*: if R is reflexive and transitive.
3. *tolerance*: if R is reflexive and symmetric.

So pre-order and tolerance are short of an equivalence relation, as they do not enforce symmetry and transitivity, respectively.

For any binary relation R on the set X , there are two induced binary relations on X , denoted as I and \simeq (we could have used a subscript R to indicate the induced nature of these relations):

1. Indifference relation I : xIy iff $\neg(xRy)$ and $\neg(yRx)$;
2. Indistinguishable relation \simeq : $x \simeq y$ iff for any $z \in X$, zRx and zRy mutually imply each other, and xRz and yRz mutually imply each other.

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Regardless of the properties (or lack thereof) of R , it can be easily seen that the two induced relations are always symmetric. The binary relation \simeq describes a relation of two elements with respect to all other elements of the set. It is always reflexive, always transitive, and hence is always an equivalence relation. On the other hand, I is reflexive iff R is irreflexive (i.e., $\neg(aRa), \forall a$), and I may still be intransitive despite of transitivity in R . The importance of intransitivity of I for behavioral science was first recognized in the invention of the semi-order (Luce, 1956).

For a binary relation (X, R) , we can define the upsets and downsets of X , as members of $\mathcal{P}(X)$, the power set of X . An upset is any set $U \in \mathcal{P}(X)$ such that if $x \in U$, then $y \in U$ for any y that satisfies xRy ; a downset is any set $D \in \mathcal{P}(X)$ such that if $x \in D$, then $y \in D$ for any y that satisfies yRx . For a given X , we denote all upsets as $U(X)$ (called “upset system”) and all downsets as $D(X)$ (called “downset system”). They are examples of subset systems.

Upsets and downsets enjoy the following well-known property:

Lemma 1. *Arbitrary union and/or intersection of upsets is an upset. Arbitrary union and/or intersection of downsets is a downset.*

Hence, for any binary relation R , the associated upset system $U(X)$ forms a complete lattice, with \wedge implemented as set-intersection (\cap) and \vee as set-union (\cup) operations on elements of power-set $\mathcal{P}(X)$, that is, $\forall u_1, u_2 \in U(X)$:

1. \wedge -operation: $u_1 \wedge u_2 = \{x \in X \mid x \in u_1 \text{ and } x \in u_2\}$;
2. \vee -operation: $u_1 \vee u_2 = \{x \in X \mid x \in u_1 \text{ and } x \in u_2\}$.

Hence $U(X)$ provides a “model” of the original binary relation (X, R) : on the one hand, they are isomorphic with R prescribed on X and used to construct $U(X)$ in the first place (see next Proposition); on the other hand, $(U(X), \wedge, \vee)$ forms a lattice, whereas (X, R) is generally not a lattice. The above discussions apply to downset system $D(X)$ as well.

Now let us consider the following sets:

$$U(x) = \{y \in X \mid xRy\}$$

$$D(x) = \{y \in X \mid yRx\}.$$

They are used as equivalent definitions of the binary relation R :

$$xRy \Leftrightarrow y \in U(x) \Leftrightarrow x \in D(y).$$

It is obvious that R is reflexive iff x is contained within $U(x)$: $x \in U(x)$.

Proposition 5. *With respect to a binary relation (X, R) , let $U(x)$ and $D(x)$ be defined as above. Consider the following three statements:*

- (i) xRy ;
- (ii) $U(y) \subset U(x)$;
- (iii) $D(x) \subset D(y)$.

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Then:

1. When R is transitive, then (i) implies (ii) and (iii);
2. When R is reflexive, then (ii) or (iii) implies (i);
3. When R is a pre-order, then (i), (ii), and (iii) are equivalent.
4. When R is symmetric, then $U(x) = D(x)$ for all $x \in X$.

When R is transitive, then $U(x)$ are upsets and $D(x)$ are downsets themselves. The mapping $x \mapsto U(x)$, when viewed as a map from X to $\mathcal{P}(X)$, preserves the order relation of elements in X ; the same holds for the mapping $x \mapsto D(x)$. Furthermore, $U(x)$ (or $D(x)$) are always join-irreducible nodes of the lattice $U(X)$ (or the lattice $D(X)$), for all $x \in X$.

4.2.2 Alexandrov Topology and Pre-order

Because of the important status of Alexandrov topology (a subset system itself) with respect to a generic subset system, we review background materials relating pre-order to Alexandrov topology; in particular, we review the notion of “specialization order.”

Definition 15. A topology \mathcal{F} is an Alexandrov topology if an arbitrary intersection of open sets is open.

A set-theoretic complement of an open set is called a closed set. In Alexandrov topology, by definition, open sets and closed sets have “symmetrical” standing.

A topological space (X, \mathcal{F}) , i.e., a set X equipped with a topology \mathcal{F} , has a well-defined *topological closure* operation on any subset A of X : \bar{A} is the intersection of all closed sets that contain A ; it is the smallest closed set that contains A . Using this topological closure operation, we can define a binary relation \lesssim on X :

$$x \lesssim y \text{ iff } \overline{\{x\}} \subset \overline{\{y\}}.$$

It can be shown that this definition is equivalent to saying that, $x \lesssim y$ iff for any open set U , $x \in U$ implies that $y \in U$. So open set here plays the role of upset for the introduced order $x \lesssim y$.

Because the binary relation \lesssim on X is defined by a set-inclusion relationship, (X, \lesssim) is a pre-order, called the *specialization pre-order* on X , induced from the topology \mathcal{F} on the same set X .

As a consequence of Lemma 1, the set $U(X)$ of all upsets of binary relation (X, R) form an Alexandrov topology on the set X , which is called the Alexandrov upset topology \mathcal{F}_R of the relation (X, R) . Under this topology, open sets are identified with upsets, and the downsets can be shown to be equivalent to the closed sets. In particular, assuming R is transitive, the lower holdings $D[x] = D(x) \cup \{x\}$ equals the closure of the singleton set $\{x\}$: $D[x] = \overline{\{x\}}$. Furthermore, $\{U[x] : x \in X\}$ is the topological base of

\mathcal{F}_R because $U[x]$ is the smallest open neighborhood of x . For any x , $U(x)$ is the deleted open neighborhood of x if R is irreflexive. The existence of the smallest neighborhood makes the verification of the accumulation point much easier. It can be shown that under \mathcal{F}_R , x is the accumulation point of a set A iff $(U(x)\setminus\{x\})\cap A \neq \emptyset$.

It is a well-known result that any Alexandrov topology is in one-to-one correspondence with pre-order:

Lemma 2 (Correspondence between Alexandrov topology and pre-order). *Let X be a set, with P denoting a pre-order on X , and \mathcal{F} denoting an Alexandrov topology on X .*

1. *Start from any pre-order (X, P) , with its Alexandrov upset topology denoted by (X, \mathcal{F}_P) . Then the specialization pre-order \lesssim of \mathcal{F}_P is the same as P .*
2. *Start from any Alexandrov topology (X, \mathcal{F}) , with its specialization pre-order denoted by (X, \lesssim) . Then the Alexandrov upset topology \mathcal{F}_{\lesssim} with respect to \lesssim (i.e., constructed by treating upsets of \lesssim as open sets) is the same as \mathcal{F} .*

Proof. For part 1, we only need to show that $x \lesssim y$ implies that xPy . Notice that $U[x]$ is an upper set. Then $y \in U[x]$ implies that xPy . For part 2, first we show that $\mathcal{F} \subset \mathcal{F}_P$. Assume $A \in \mathcal{F}$. Then for any $x \in A$, xPy implies that $y \in A$. Hence A is an upper set with respect to P . Then we show that $\mathcal{F}_P \subset \mathcal{F}$. Only need to show that $U[x] \in \mathcal{F}$. Assume $A \in \mathcal{F}$ and $x \in A$. For any $y \in U[x]$, we have $y \in A$. Hence, $U[x]$ is the intersection of all open neighborhoods of x which is still open by the property of Alexandrov topology.

It is also known that Alexandrov topology is the finest topology that enjoys this one-to-one correspondence to a given pre-order.

4.3 PRE-ORDER AND TOLERANCE ON V INDUCED FROM (V, E)

In this section, we start from a general subset system (V, E) and derive its properties. To simplify our discussion, we assume that in a subset system (V, E) , E is a simple set and every element v in V appears in some e in E . No other assumptions about the make-up of E is assumed other than that it is a collection of subsets of V .

Recall from the Introduction that a subset system (V, E) has a natural pre-order \preceq among its members (subsets of V) using the set-inclusion relation:

$$e_1 \preceq e_2 \text{ (as members of } E) \leftrightarrow e_1 \subset e_2 \text{ (as subsets of } V).$$

This is the order among members of E . Likewise, (V, E) also has a natural tolerance \preceq among its members (that is, members of E) using the set-

intersection relation:

$$e_1 \simeq e_2 \text{ (as members of } E) \leftrightarrow e_1 \cap e_2 \neq \emptyset \text{ (as subsets of } V).$$

Both \preceq and \simeq defined on members of E are inherited from the natural pre-order and tolerance on the power-set $\mathcal{P}(V)$. The remainder of this section will discuss two new relations, a pre-order P and a tolerance T , defined on elements of V as induced from the subset system (V, E) .

4.3.1 Neighborhood and Distinguishability

Each member of $e \in E$ for which $v \in e$ is called a *neighbor* of v . Denote the collection of members of E (each of which is a subset of V) containing the element v by $E(v)$. For every v , $E(v)$ is a sub-collection of E , called the *neighborhood system* of v . Denote $\mathcal{P}\mathcal{P}(V)$ as the power-set of $\mathcal{P}(V)$, then $E : V \rightarrow \mathcal{P}\mathcal{P}(V)$ in fact defines a mapping, such that (E, ε) , where

$$\varepsilon = \{E(v) \mid v \in V\},$$

is a subset system on E (i.e., with members of E as base set).

With the notion of neighbor, we may define various degrees of pair-wise separability (distinguishability) for elements in V . Two points v_1, v_2 are called *S-indistinguishable* iff all neighbors of v_1 are neighbors of v_2 , and vice versa. In other words, v_1 and v_2 are *S-indistinguishable* iff there is no subset e in E that contains one but not the other. When two elements v_1, v_2 are not *S-indistinguishable*, then they are said to be *S-distinguishable*. *S-indistinguishability* can be shown to be an equivalence relation.

Given two elements v_1, v_2 , if v_1 has a neighbor which does not contain v_2 , and vice versa for v_2 , then v_1, v_2 are said to be *S₁-distinguishable*. Otherwise, when no such neighbor exists, these two elements are *S₁-indistinguishable*. *S₁-distinguishability* can be shown to be symmetric but not reflexive. *S₁-indistinguishability* is a tolerance relation.

Given two elements v_1, v_2 , if there exists a neighbor e_1 of v_1 and a neighbor e_2 of v_2 such that e_1 does not contain v_2 , e_2 does not contain v_1 , and $e_1 \cap e_2 = \emptyset$, then v_1, v_2 are said to be *S₂-distinguishable*. Otherwise, when no such pair of neighbors exist, these two elements are *S₂-indistinguishable*. *S₂-distinguishability* can be shown to be a tolerance relation.

Formally, these definitions are written:

1. *S-indistinguishable*: $E(v_1) = E(v_2)$;
2. *S-distinguishable*: $E(v_1) \neq E(v_2)$;
3. *S₁-distinguishable*: there exist $e_1 \in E(v_1), e_2 \in E(v_2)$ such that $v_2 \notin e_1, v_1 \notin e_2, e_1 \cap e_2 \neq \emptyset$;
4. *S₂-distinguishable*: there exist $e_1 \in E(v_1), e_2 \in E(v_2)$ such that $v_2 \notin e_1, v_1 \notin e_2, e_1 \cap e_2 = \emptyset$;

4.3.2 Induced Pre-order on V

Given (V, E) , we can define a pre-order P among elements of V , called *specialization order*, as follows.

Definition 16. vPu if and only if $E(v) \subset E(u)$.

It is easy to show that P is transitive and reflexive, and hence, is a pre-order. We call P a specialization pre-order because it is a generalization of the terminology used in the setting of a topology: when the subset system is a topology of V , then P is just the specialization pre-order with respect to the topology. See Section 2.2.

One immediate application of this pre-order P is its relationship to the various degrees of distinguishability we introduced earlier.

Theorem 4. Given a subset system (V, E) , we have the following statements:

1. vPu and uPv if and only if $E(v) = E(u)$, i.e., they are S -indistinguishable;
2. vPu and $\neg(uPv)$ if and only if $E(v) \subsetneq E(u)$, i.e., v and u are S -distinguishable, but not S_1 -distinguishable;
3. $\neg(vPu)$ and $\neg(uPv)$ if and only if $E(v) \not\subset E(u)$ and $E(u) \not\subset E(v)$, i.e., v and u are S_1 -distinguishable.

Also notice that \subset is a pre-order on E , and it plays the same role as P on V .

From the P induced from (V, E) , we can construct the set of upsets $U(V)$ and the set of downsets $D(V)$, which are subset systems themselves. Note the differences of U and D from E , all of which are subset systems, that is, distinct points of $\mathcal{P}\mathcal{P}(V)$. As sets, U and D have equal cardinality, which is equal to the cardinality of V . On the other hand, E as a set may have larger or smaller cardinality.

Recall (Section 2.1) upset $U(v)$ (and downset $D(v)$) of any element $v \in V$. The proposition below gives a precise relationship between $U(v)$ and $E(v)$.

Proposition 6. $U(v) = \cap E(v)$.

Proof. First we show that $U(v) \subset e$ for every $e \in E(v)$. For any $u \in U(v)$, $E(v) \subset E(u)$. So $u \in e$ for any $e \in E(v)$, i.e., $u \in \cap E(v)$. On the other hand, we want to show that $\cap E(v) \subset U(v)$. For any $u \in \cap E(v)$, every member in $E(v)$ contains u , hence $E(v) \supset E(u)$, that is, vPu . So $u \in U(v)$.

With respect to the Alexandrov upset topology of (V, P) , each $e \in E(v)$ is a neighborhood (in a topological sense) of v , and is open.

Proposition 7. Every e in E is an open set with respect to the Alexandrov upset topology determined by P .

Proof. For every $v \in e$, $U(v) \subset e$ by Proposition 6. Hence e must be an upset with respect to P .

When the subset system (V, E) forms an Alexandrov topology, $E = U(V)$, then $\{U(v) \mid v \in V\}$ forms the collection of the local basis of the Alexandrov upset topology $U(V)$.

4.3.3 Clique, Block, Circuit, and Atom

With respect to a tolerance relation T on X , a *clique* is a subset of elements of X such that they are all pairwise in tolerance relations; this subset of elements forms an equivalence class. A *block* is a maximal clique; that is, no additional element can be added to still make the subset a clique (i.e., all pairwise in tolerance relations). A *circuit* is a subset of X that is not a clique and is minimally so; that is, removing any of its elements turns it into a block.

Lemma 3. *Any subset of a clique is a clique (“hereditary property”). The set of all cliques, which is a subset system, is closed under set-wise intersection operation.*

Note that the union of two cliques is, in general, no longer a clique.

Denote $T(x) = \{y \mid xTy\}$, the set of elements that are in tolerance relation T with x ; we call it an *adjacent set*. $T(x)$ has a simple graph-theoretic interpretation: it is the set of neighboring (adjacent) vertices connected to vertex x . Any clique involving $x \in X$ must be a subset of $T(x)$. Stated differently, for any x , if the set $T(x)$ forms a clique, then it must be a block.

For a tolerance relation, some blocks have special elements which are called “atoms.” We denote $B(x)$ as those elements of X that may participate in a specific block B containing the element x . Note that given x , $B(x)$ as a subset of X may not be unique. Clearly, $B(x) \subseteq T(x)$. If, for certain a , we have $B(a) = T(a)$, then a is called an “atom,” or generic point, of the block $B = T(a)$. **Formally,**

Definition 17. *For a block B under tolerance T , any element $a \in B$ such that $T(a) = B$ is called an atom of B .*

It can be shown that any atom can only belong to one block. We have the following characterization for the “atomic” property of an element.

Proposition 8. *An element a in X is an atom of some block $B = T(a)$ with respect to the tolerance T if and only if $\cap T(T(a)) = T(a)$. Here, $T(A)$ is defined as follows for any subset A of X .*

$$T(A) = \{T(a) \mid a \in A\}.$$

A tolerance T on X is called *connected* if for any elements $x, y \in X$, there exist elements $a_1, \dots, a_n \in X$ such that $xTa_1Ta_2 \dots Ta_nTy$; in this case X is called T -connected.

Recall that a covering C on a set X is a collection of subsets of X whose union is X :

$$\bigcup_{e \in C} e = X.$$

As is easily shown, the collection of all blocks of a tolerance T on a set X form a covering of X if X is T -connected. Conversely, given a covering C as a subset system of X , we define a tolerance relation T by declaring xTy iff there is a member $e \in C$ such that $x \in e, y \in e$. Then T is necessarily

connected on X . If C is a non-redundant cover, meaning that the removal of any $e \in C$ will destroy the “covering” property, then each member of C is a block with respect to the induced T . Hence, each member of a non-redundant cover contains some atoms.

4.3.4 Induced Tolerance on V

With respect to any subset system (V, E) , we now construct a tolerance relation T on V . Recall that $E(v)$ denotes the subcollection of E which contains the element v .

Definition 18. vTu if and only if $E(v) \cap E(u) \neq \emptyset$.

This definition is equivalent to the statement: vTu if and only if there exists $e \in E$ such that $v \in e$ and $u \in e$.

In hypergraph language, what we have done is to construct a new (regular) graph on the original set of nodes, such that two nodes are (undirectedly) connected if in the original hypergraph there is at least one hyper-edge connecting them. Comparing their respective definitions, Definition 16 and Definition 18, we easily have:

Proposition 9. vPu implies that vTu .

Lemma 4. For subset system (V, E) , every $e \in E$ is a clique with respect to the induced tolerance relation T given by Definition 18.

Furthermore, we can verify that

Proposition 10. $T(v) = \cup E(v)$.

The following theorem gives a sufficient condition for $e \in E$ to be a block.

Theorem 5. For $e \in E$, if $e \cap e' \neq \emptyset$ implies that $e' \subset e$ for any $e' \in E$, then e is a block.

Proof. To show that e is a block, we only need to show that if there is any $v \in V$ that vTu for any $u \in e$, then $v \in e$. If there is such v , then we have $e \subset T(v)$. If $e \in E(v)$, then we have $v \in e$. If $e \notin E(v)$, then for any $u \in e$, there must be some $e' \in E(v)$ such that $u \in e'$, which implies that $e' \subset e$. But notice that $v \in e'$, so we get $v \in e$.

When a set of non-redundant blocks covers V , then each block contains at least one atom. Using the pre-order P induced from the same subset system, then every element can be put into an ordered set, with atoms at the bottom.

4.3.5 Subset Systems as Captured by P and T

The main results of the previous two sections can be summarized as:

Theorem 6. Let (V, E) be a subset system, with neighborhood function $E(v)$ for any $v \in V$. Denote $U(v)$ as the upset and $T(v)$ as the adjacent set, corresponding to the induced pre-order P and tolerance T , respectively. Then

1. $U(v) = \cap E(v)$;
2. $T(v) = \cup E(v)$.

Defining P and T on the ground-set V of a subset system (V, E) is “natural,” if we want the collection of subsets to provide a “context” for elements of V . Let us treat each member of E as a context, as defined by the co-occurrence of certain elements of V . Then, the relationship P gives information about whether one element is more generic/special than another element, as determined by the set of contexts each element appears in. On the other hand, the relationship T gives information about whether two elements share a same context or not. Given these interpretations, it is natural to ask:

1. Independence of P and T : Are P and T two *independent* relations induced from the same subset system?
2. Information Completeness of P and T : Do P and T encode complete information about (V, E) , so that (V, E) can be recovered by them?

Below, we give negative answers to both these questions.

Regarding (1), the following are examples of subset systems sharing the same specialization pre-order P but with different tolerance T , or vice versa.

Example 4.3.1. (Same P with different T).

For $V = \{a, b, c\}$, take $E_1 = \{\{a\}, \{b\}, \{c\}\}$ and $E_2 = \{\{a, b\}, \{b, c\}, \{a, c\}\}$. (V, E_1) and (V, E_2) induce the same pre-order (a trivial one) with different tolerance.

Example 4.3.2. (Same T with different P).

For $V = \{a, b, c\}$, take $E_1 = \{\{a, b, c\}\}$ and $E_2 = \{\{a\}, \{a, b, c\}\}$. (V, E_1) and (V, E_2) induce the same tolerance with different pre-orders.

Regarding (2), the following is an example of different subset systems with the same pre-order (V, P) and the same tolerance (V, T) .

Example 4.3.3. (Same P and T , but different subset systems).

For $V = \{a, b, c, d\}$, take $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$ and $E_2 = \{\{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}\}$. (V, E_1) and (V, E_2) induce the same tolerance and specialization pre-order on V .

To conclude, P and T are two independent relations induced from the same subset system. By independence, we mean there exist subset systems sharing the same specialization pre-order P but with different tolerance T , or vice versa. Furthermore, different subset systems, say (V, E) and (V, E') , can induce the same pre-order P on V .

4.3.6 Galois Connection Between V and E

Given any subset system (V, E) , we can define two operations as follows:

$$A^\triangleright = \{e \in E \mid A \subset e\}$$

$$F^\triangleleft = \left\{ v \in V \mid \bigcap_{e \in F} e \right\}$$

Then $\triangleright : \mathcal{P}(V) \rightarrow \mathcal{P}(E)$ and $\triangleleft : \mathcal{P}(E) \rightarrow \mathcal{P}(V)$ form a pair of order-reversing maps between $\mathcal{P}(V)$ and $\mathcal{P}(E)$; that is, the following holds for any $A \subseteq V$, $F \subseteq E$:

$$A^\triangleright \subseteq F \leftrightarrow A \supseteq F^\triangleleft.$$

The pair of maps \triangleright and \triangleleft are called (antitone version of) *Galois connection*; they enjoy nice properties:

1. $A_1 \subseteq A_2 \rightarrow A_1^\triangleright \supseteq A_2^\triangleright$ and $F_1 \subseteq F_2 \rightarrow F_1^\triangleleft \supseteq F_2^\triangleleft$;
2. $A^\triangleright = A^{\triangleright \triangleleft}$ and $F^\triangleleft = F^{\triangleleft \triangleright}$;
3. $A \subseteq A^{\triangleleft \triangleright}$ and $F \subseteq F^{\triangleright \triangleleft}$.

The pair of maps are adjoints of each other (exchanging the domain and target spaces). Each of \triangleright and \triangleleft uniquely determines the other, via the formulae:

$$A^\triangleright = \inf \{F \subseteq E \mid A \supseteq F^\triangleleft\}$$

$$F^\triangleleft = \inf \{A \subseteq V \mid A^\triangleright \subseteq F\}$$

The statement “ \triangleright is a surjective map” is equivalent to “ \triangleleft is an injective map,” which is equivalent to $\triangleright \triangleleft = \text{Id}$. An analogous set of equivalent statements can be obtained by switching \triangleright and \triangleleft in the above.

The compositions $\text{Cl}_V \equiv \triangleright \triangleleft$ and $\text{Cl}_E \equiv \triangleleft \triangleright$ are monotonic and idempotent maps on $\mathcal{P}(V)$ and on $\mathcal{P}(E)$, respectively:

1. Monotonicity: $\text{Cl}_V(A_1) \subseteq \text{Cl}_V(A_2)$, $\forall A_1 \subseteq A_2 \subseteq V$ and $\text{Cl}_E(F_1) \subseteq \text{Cl}_E(F_2)$, $\forall F_1 \subseteq F_2 \subseteq E$;
2. Idempotency: $\text{Cl}_V(\text{Cl}_V(A)) = \text{Cl}_V(A)$ and $\text{Cl}_E(\text{Cl}_E(F)) = \text{Cl}_E(F)$.

Together with the *non-contraction* property (3) above, Cl_V and Cl_E are, by definition, closure operators on $\mathcal{P}(V)$ and on $\mathcal{P}(E)$. The fixed points of these operators form the so-called “*concept lattices*” (Ganter & Wille, 2012). In such lattices, \triangleright preserves existing joins, and \triangleleft preserves existing meets.

Dual to the closure operator Cl is the interior operator Int , which is also monotonic and idempotent. Denote $A^c = X \setminus A$ for any $A \subseteq X$. Then $\text{Int}(A) = (\text{Cl}(A^c))^c$ and $\text{Cl}(A) = (\text{Int}(A^c))^c$. Unlike the *non-contracting* property of Cl , Int is *non-expanding*.

4.3.7 Upper and Lower Approximations on V

The two induced relations P and T on the ground-set V allow us to define rough approximations of any subsets of V .

Let R be any binary relation on V , with $U(v) = \{w \in V : vRw\}$ for any $v \in V$. For any subset A , the lower approximation $\blacktriangledown : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ and

upper approximation $\blacktriangle : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ of A with respect to the binary relation R :

$$\begin{aligned} A^\blacktriangledown &= \{v \in V \mid U(v) \subseteq A\}, \\ A^\blacktriangle &= \{v \in V \mid U(v) \cap A \neq \emptyset\}. \end{aligned}$$

The operation \blacktriangle preserves \cup , whereas \blacktriangledown preserves \cap . They are both order-preserving; that is, $X \subseteq Y$ implies $X^\blacktriangledown \subseteq Y^\blacktriangledown$ and $X^\blacktriangle \subseteq Y^\blacktriangle$.

Let A^c denote the complement $\forall A$ of A . Then:

$$A^{\blacktriangle c} = A^{c\blacktriangledown}, \quad A^{\blacktriangledown c} = A^{c\blacktriangle}.$$

This means, \blacktriangle and \blacktriangledown are dual.

The properties of the binary relation R leads to the following properties of lower and upper approximations (where $A \subseteq V$, i.e., $A \in \mathcal{P}(V)$):

- when R is reflexive: $A^\blacktriangledown \subseteq A \subseteq A^\blacktriangle$;
- when R is symmetric: $A^{\blacktriangledown\blacktriangle} \subseteq A \subseteq A^{\blacktriangle\blacktriangledown}$;
- when R is transitive: $A^{\blacktriangle\blacktriangle} \subseteq A^\blacktriangle$ and $A^\blacktriangledown \subseteq A^{\blacktriangledown\blacktriangledown}$;
- when R is left-total: $A^\blacktriangledown \subseteq A^\blacktriangle$.

We may also determine rough set approximations using the downset system $D(v)$:

$$\begin{aligned} A^\blacktriangledown &= \{v \in V : D(v) \subseteq A\}, \\ A^\blacktriangle &= \{v \in V : D(v) \cap A \neq \emptyset\}. \end{aligned}$$

The pair $(\blacktriangle, \blacktriangledown)$ and the pair $(\blacktriangledown, \blacktriangle)$ are both order-preserving Galois connections on $\mathcal{P}(V)$.

When R is a pre-order, then for $A \subseteq V$:

$$A^{\blacktriangle\blacktriangle} = A^\blacktriangle, \quad A^{\blacktriangledown\blacktriangledown} = A^\blacktriangledown, \quad A^{\blacktriangle\blacktriangledown} = A^\blacktriangle, \quad A^{\blacktriangledown\blacktriangle} = A^\blacktriangledown.$$

This is to say, \blacktriangle and \blacktriangle are closure operators, and \blacktriangledown and \blacktriangledown are interior operators. Moreover,

$$A^{\blacktriangle\blacktriangledown} = A^\blacktriangle, \quad A^{\blacktriangledown\blacktriangle} = A^\blacktriangledown, \quad A^{\blacktriangledown\blacktriangle} = A^\blacktriangledown, \quad A^{\blacktriangle\blacktriangledown} = A^\blacktriangle.$$

These two approximations determine two Alexandrov topologies on V :

$$\mathcal{T}^\uparrow = \{A^\blacktriangle : A \subseteq V\} = \{A^\blacktriangledown : A \subseteq V\}$$

and

$$\mathcal{T}^\downarrow = \{A^\blacktriangledown : A \subseteq V\} = \{A^\blacktriangle : A \subseteq V\}$$

These topologies are dual; that is,

- $A \in \mathcal{F}^\uparrow \Leftrightarrow A^c \in \mathcal{F}^\downarrow$
- Under \mathcal{F}^\uparrow , \blacktriangle is the closure operator and ∇ is the interior operator; \blacktriangle is the smallest neighborhood operator, and the set $\{\{v\}^\blacktriangle : v \in V\} = \{U(v) : v \in V\}$ is the smallest base.
- Under \mathcal{F}^\downarrow , \blacktriangle is the closure operator and \blacktriangledown is the interior operator; \blacktriangle is the smallest neighborhood operator, and the set $\{\{v\}^\blacktriangle : v \in V\} = \{D(v) : v \in V\}$ is the smallest base.
- We have:

$$\{v\}^\blacktriangledown = \emptyset, \quad \{v\}^\nabla = \emptyset,$$

$$\{v\}^\blacktriangle = D(v), \quad \{v\}^\blacktriangle = U(v).$$

When R is tolerance, $\blacktriangle = \triangle$ and $\blacktriangledown = \nabla$, so

$$A^\blacktriangle = A^\triangle, \quad A^\blacktriangledown = A^\nabla.$$

for any $A \subseteq V$. In this case, $\blacktriangle : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ is a monotone Galois connection on $\mathcal{P}(V)$. The map $A \mapsto A^\blacktriangledown$ is a closure operator, with the set $\{A^\blacktriangledown \mid A \subseteq V\}$ as fixed points. The map $A \mapsto A^\blacktriangle$ is the interior operator, with the set $\{A^\blacktriangle \mid A \subseteq V\}$ as fixed points. We have:

$$\{v\}^\blacktriangledown = \{v\}^\nabla = \emptyset,$$

$$\{v\}^\blacktriangle = \{v\}^\triangle = T(v).$$

4.3.8 Complement of Subset System

Given a subset system (V, E) , we can also consider the complement system (V, E^{co}) , where $E^{co} = \{\forall e \mid e \in E\}$. It is called a complement system because if we denote the relation defined by (V, E) as R , then the relation R^c can be defined as $vR^c e$ if and only if $v \in \forall e$. Here, we need another assumption that $\cap E = \emptyset$, so that every v is contained in some member of E^{co} . The system (V, E^{co}) also gives a pre-order and a tolerance on V , denoted by P^{co} and T^{co} respectively. As we defined above, $v_1 P v_2$ if and only if $E(v_1) \subset E(v_2)$. And this is equivalent to $E^{co}(v_1) \supset E^{co}(v_2)$, which is further equivalent to $v_2 P^{co} v_1$. Therefore, we have the following proposition.

Proposition 11. $v P u$ if and only if $u P^{co} v$.

Different from the relation between P and P^{co} , T and T^{co} encode non-redundant information of the system. Neither one of them implies the other, because $v_1 T v_2$ if and only if there is an $e \in E$ such that $v_1 \in e$, $v_2 \in e$, and $v_1 T^{co} v_2$ if and only if there is an $e \in E$ such that $v_1 \notin e$, $v_2 \notin e$. It is possible that $v_1 T v_2$ and $v_1 T^{co} v_2$ both hold. Therefore, Proposition 9 can be strengthened:

Proposition 12. $v P u$ implies both $v T u$ and $v T^{co} u$.

4.3.9 Chained Subset System

We call a subset system (V, E) “chained” if $\{E(v), v \in V\}$, when properly arranged, forms an increasing (or decreasing) sequence of sets. When a subset system (V, E) is chained, then the corresponding cross-table can be shown to be a “biorder” (Guttman scale). In particular, both (V, P) and (E, \preceq) are weak orders; they are linearly ordered by inclusion relation. This leads to the following two propositions characterizing a chained subset system.

Proposition 13. *Let I denote the indifference relation with respect to the specialization pre-order P induced by (V, E) , that is, vIu iff neither vPu nor uPv . Then we have the conclusion: I is empty if and only if the subset system (V, E) is chained.*

Using the well-known characterization of biorders (Doignon, Ducamp, & Falmagne, 1984), we have:

Proposition 14. *A subset system (V, E) is a chained system iff the following holds for all $v_1, v_2 \in V$ and $e_1, e_2 \in E$:*

If $v_1 \in e_1, v_2 \in e_2$, then either $v_1 \in e_2$ or $v_2 \in e_1$.

4.3.10 Subset Systems with Same P

We already see that \preceq and P are two different pre-orders, on E and on V respectively, associated with any subset system (V, E) . Whereas \preceq is naturally endowed, P depends on the collection E of subsets. We now investigate various subset systems that will yield the same specialization pre-order P .

First, recall Lemma 2 stating that a given pre-order P uniquely corresponds with an Alexandrov topology, which is a subset system itself. This means that if the subset system (V, E) we started with is already an Alexandrov topology (i.e., E is already closed under arbitrary union and arbitrary intersection), then we cannot add more members to E without perturbing (V, P) as a pre-order.

For any subset system (V, E) , when fixing V , we perform two incremental steps of closure to E , to turn (V, E) into (1) topped \cap system, and then into (2) Alexandrov topology. During each step, the induced pre-order on V remains unchanged.

In the first step, we supply more members to E to turn E from a pre-ordered set (when E is viewed as a point in $\mathcal{P}\mathcal{P}(V)$) into a complete lattice E' of sets (where each member of E' is a subset of V). These additional members e 's are of the form: $e = \cap e_i$, where each e_i is an member of E . Through this operation, we make \wedge operation on elements of E identical with \cap , the set-wise intersection operation on subsets of V . The resulting E' forms a complete lattice (because any topped intersection system is a complete lattice; see Davey & Priestley, 2002).

In the second step, we supply more members to E' to turn E' from a complete lattice into complete distributive lattice E'' of sets. The additional

Proof

members e' are of the form: $e = \cup e'_i$, where each e'_i is a member of $E' \supset E$. Through this operation, we make \vee operation on members of E identical with \cup , the set-wise union operation on subsets of V . The subset system (V, E'') is an Alexandrov topology \mathcal{T}_E , whereby each member $e \in E''$ is its open set of \mathcal{T}_E . In particular, each $e \in E$ is an open set (Proposition 7 proved earlier). Once in E'' , its specialization pre-order P is the specialization pre-order \preceq of the Alexandrov topology, as given by Lemma 2.

$$\begin{array}{lcl} (V, E) & \rightarrow & (V, E') \quad \rightarrow \quad (V, E'') \equiv (V, U(V)). \\ \text{Subset System} & \rightarrow & \text{Topped } \cap \text{ System} \quad \rightarrow \quad \text{Alexandrov topology.} \end{array}$$

4.4 DISCUSSION

In this chapter, we proposed the notion of *subset system* (V, E) as a set-theoretic foundation of relational structure. This conceptualization merely stipulates that whenever a set V is specified, one needs to also specify the collection of subsets E . These two aspects are inalienably linked, in order for elements of V , as well as for members of E , to be in various relations with each other. So our conceptualization for binary relations refocuses from *set* to *systems of subsets*.

We showed that any subset system (V, E) comes with

1. two natural relations on the set E :

- pre-order \preceq
- tolerance \preceq

2. two induced relations on the set V :

- pre-order P
- tolerance T

Therefore, subset system (V, E) provides the concise mathematical language to describe the “context” (modeled by E) for relationships between objects (modeled by V). This framework, thus, may provide rigorous yet rich vocabulary for defining part-whole relations, context, and configural/Gestalt processes as investigated by the honoree of this book, Prof. J. T. Townsend (e.g., Eidels, Townsend, & Pomerantz, 2008; Wenger & Townsend, 2001).

Readers familiar with elementary point-set topology (e.g., Pervin, 2004) will notice that the notions introduced in this chapter for neighborhood, (various levels of) distinguishability, and the induced order P , etc., mirror corresponding notions in topology (topological neighborhood, topological distinguishability, and separability, specialization pre-order). Our analysis

shows that these concepts can be defined independently of the closure requirement of the subset system discussed in Section 4.3.10

Finally, we discuss briefly how to modify subset systems and introduce dynamics through learning. Given V , the binary relations P and T on its elements are completely specified whenever E is specified. There are two operations that can be implemented to modify a subset system, one is to enlarge E to E' ; this is what we discussed in Section 4.3.10. Another is to enlarge V to V' without changing the specialization pre-order among elements in V . These two operations correspond to the operations of adding attributes and adding objects into a cross-table. Future research will be devoted to understanding mechanisms through which subset systems can be “evolved” with both V and E changing.

ACKNOWLEDGEMENTS

This work is supported by ARO grant W911NF-12-1-0163 (PI: Jun Zhang).

REFERENCES

- Davey, B. A., & Priestley, H. A. (2002). *Introduction to lattices and order*. Cambridge University Press.
- Doignon, J.-P., Ducamp, A., & Falmagne, J.-C. (1984). On realizable biorders and the biorder dimension of a relation. *Journal of Mathematical Psychology*, 28(1), 73–109.
- Eidels, A., Townsend, J. T., & Pomerantz, J. R. (2008). Where similarity beats redundancy: The importance of context, higher order similarity, and response assignment. *Journal of Experimental Psychology: Human Perception and Performance*, 34(6), 1441–1463.
- Ganter, B., & Wille, R. (2012). *Formal concept analysis: mathematical foundations*. Springer Science & Business Media.
- Luce, R. D. (1956). Semiorders and a theory of utility discrimination. *Econometrica: Journal of the Econometric Society*, 24, 178–191.
- Pervin, W. J. (2004). *Foundations of general topology*. New York: Academic Press.
- Wenger, M. J., & Townsend, J. T. (2001). *Computational, geometric, and process perspectives on facial cognition: Contexts and challenges*. Mahwah, NJ: Lawrence Erlbaum Associates.

Proof

5 Uniqueness of a Multinomial Processing Tree Constructed by Knowing Which Pairs of Processes are Ordered

Richard Schweickert and Hye Joo Han

Suppose we could figure out which mental processes are executed one after another and which are not. What would we learn about the overall arrangement of the processes? Here we consider the implications for learning the structure of a multinomial processing tree underlying a task. We learn a lot: whether a tree is possible, if so, its form, and whether more than one tree is possible.

The importance of the problem of distinguishing serial and parallel processes is indicated by the continual attention James T. Townsend has devoted to it, from early work criticizing methods popular but unsound (e.g., Townsend, 1972) to recent work presenting methods sophisticated and sound (Yang, Fifić & Townsend, 2014). There has been much progress through the work of Jim and others. There are far too many papers to mention; for surveys, see Townsend and Ashby (1983), Townsend and Wenger (2004), Logan (2002), and Schweickert, Fisher and Sung (2012). The story is still unfolding.

Multinomial processing trees have been successfully used to model processes in many tasks, including perception (e.g., Ashby, Prinzmetal, Ivry, & Maddox, 1996), memory (e.g., Batchelder & Riefer, 1986; Chechile & Meyer, 1976), and social cognition (e.g., Klauer & Wegener, 1998). They are useful when each mental process involved in a task has mutually exclusive outcomes. In a memory task, for example, it may be reasonable to assume a retrieval attempt is successful or not. If not, the participant guesses, and a guess is either correct or incorrect. The tree in Figure 5.1 represents these processes. The retrieval attempt is at x . If retrieval is successful, the outcome is a . If retrieval is unsuccessful, a guess is made, represented at y . The possible outcomes of the guess are represented at b and c . Using intuition, an investigator can often rather quickly sketch a plausible multinomial tree model for a task and statistical tests are often easily conducted (e.g., Stahl & Klauer, 2007). For selection of a tree based on minimum description length, see Wu, Myung and Batchelder (2010). For reviews, see Batchelder and Riefer (1999), and Erdfelder et al. (2009).

Proof