Information Fusion with Topological Event Spaces

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***Abstract*— We develop a novel information fusion scheme on topological event space, viewed as distributive lattice. We discuss the advantages of topological modeling and compare our approach to the existing Bayesian, Dempster-Shafer, and Dezert-Smarandache approaches. The proposed scheme is described in detail and illustrated with an example of fusion of 3 sensors in the presence of missing information.**

Keywords—Information Fusion, Probability, Lattice Theory, Distributive Lattice, Topology, Topological Event Space, Belief Functions, Sensor Networks, Dempster-Shafer, Proportional Conflict Redistribution, Target Classification

I. Introduction

Modern Information Fusion utilizes probability theory to model both individual sensor performance and the rules of their combination. The Dempster-Shafer theory (DST) of belief functions provides an alternative to classical probability by assigning basic probability mass to subsets of the ground set, which models the set of elementary events, as opposed to assigning to singletons in the Bayesian methods [1]. DST eliminates the need to assign a priori probabilities to the ground set. Further development of the belief function theory resulted in the Dezert-Smarandache theory (DSmT) [2], [3]. Bayesian and DST methods perform poorly in the presence of conflicting evidence [4] [5] [6], hence the introduction of several Proportional Conflict Redistribution (PCR) rules, applicable to both DST and DSmT. The advantage of DST/DSmT over Bayesian methods comes at a price of computing with an exponentially larger event space - the set of all subsets of the ground set (the powerset). The event space of DSmT is even larger as it includes not only the subsets but also all their unions and intersections. The computational load increases rapidly with the size of the problem.

This work studies information fusion and uncertainty modeling in *topological event spaces* [7], [8], [9]. One of the advantages of using topological event space is its smaller size, compared to the Boolean event space used by Bayesian and DST approaches, yet still forms a closed system for event calculus.

The topological concepts of open and closed sets, boundary and interior, provide richer modeling environments, which are suitable for rational decision making since they admit probability functions [10]. Topological event space has been used to model human decision making and to explain results of experiments involving human judgments of probabilities [9]. Our goal is to formulate a fusion framework based on topological event spaces that can be utilized in a wide variety of applications. An initial formulation of such a framework briefly reported in [11]. Here we provide detailed descriptions of the steps involved in implementing the system and illustrate a possible application of our framework to the fusion of feature detectors with missing information.

The rest of this paper is organized as follows. We provide theoretical background on topological event spaces in Sec. II. Beliefs and probabilities on topological spaces are best explained through connection with the lattice theory, and this connection is the subject of Sec. III. The background on belief functions on Boolean event spaces and arbitrary lattices is covered in Sec. IV. Based on the introduced concepts, Sec. V describes the proposed topological fusion scheme. We consider the problem of classifier fusion with missing information that can be solved with our approach in Sec. VI. Sec. VII provides the discussion of our results and future directions of our topological approach.

II. Boolean and Topological Event Spaces

Throughout this paper, we consider finite sets. A *topological space*, or just a *topology*, is an ordered pair $(\Omega, \mathcal{T})$, where Ω is a set (where each element is called a *point*), and $\mathcal{T}$ is a collection of subsets of Ω, such that $\emptyset \in \mathcal{T}, \Omega \in \mathcal{T}$, and $\mathcal{T}$ is closed under finite intersections and arbitrary unions of its elements [12]. The elements of $\mathcal{T}$ are called *open sets*. Suppose $A \in \mathcal{T}$. The set-theoretical complement of the subset A is $\neg A = \Omega \backslash A$; any subset $\neg A$ that is the complement of an open set is called a *closed* set. The collection of closed sets is also closed under unions and intersections. It is possible for a set to be closed and open at the same time (clopen), or neither. The extreme cases of topological spaces are the *discrete topology*, which consists of *all* subsets of Ω, that is the powerset 2^{Ω}, and the *indiscrete topology* consisting of only two elements: $\emptyset$ and Ω.

Designating Ω to be the set of elementary events, or the ground set, we model the set of all possible events by the topology $\mathcal{T}$. The discrete topology corresponds to the Boolean event space that is used by the classical probability.

The events in Boolean space are closed under the operations of set intersection, union, and complementation. They form a Boolean algebra of events, which directly corresponds to classical propositional logic. The topological space is closed under union and intersection, but not under complementation. It corresponds to intuitionistic logic [13]. Thus, topological modeling allows capturing the features of intuitionistic logic, with the relaxation of the law of excluded middle as its centerpiece. Evidence negating a certain event does not exclude the possibility of some additional "third" event occurring. In addition to capturing intuitionistic logic, topological spaces are also suitable for probabilistic reasoning. In order to consider probabilities on topological event spaces we need to briefly discuss the connection between topology and lattice theory.

III. LATTICE THEORY AND TOPOLOGICAL SPACES

A comprehensive treatment of Lattice Theory can be found in [14] and here we only provide the basic definitions. A *partial order* on a set P is a binary relation, denoted by $\leq$ that is *reflexive*, *antisymmetric*, and *transitive*. Sets with partial order are called *posets*. Given a subset $S \subseteq P$, an element $x \in P$ is an *upper bound* of S if $(\forall s \in S), s \leq x$. The lower bound is defined analogously. The *least upper bound* is an element $x \in P$ such that x is an upper bound of S , and for any other upper bound y of S, $x \leq y$. The *greatest lower bound* is defined analogously.

Given any two elements, $x, y \in P$, we denote their *least upper bound* and dually, *greatest lower bound*, by $x \vee y$ and $x \wedge y$. These operations are respectively called *join* and *meet*.

A partially ordered set P such that join and meet exist for all pairs of its elements is called a *lattice*. The lattice is called *complete* if meet and join exist for any subset of P.

A finite lattice always has the least element, or *bottom*, denoted by 0, and the greatest element (*top*), denoted by 1. An important operation on a lattice is that of *complementation*. For any element $a \in P$, its complement is an element $a' \in P$, such that $a' \wedge a = 0$ and $a' \vee a = 1$. Note that for an arbitrary element of P, its complement may not exist, or it may have multiple complements. Finite lattice is often visualized as a Hasse diagram [14], for example see Fig.1.

A lattice is called *distributive* if for any lattice elements a, b, c the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ is satisfied. In a distributive lattice, it is known that an element can have at most one complement.

Lattice of sets is a lattice formed by a family of subsets of the powerset of Ω , $P \subseteq 2^{\Omega}$, with the binary relation corresponding to set inclusion: for all $\mathrm{A}, \mathrm{B} \in \Omega, \mathrm{A} \leq \mathrm{B} \Leftrightarrow \mathrm{A} \subseteq \mathrm{B}$ and the meet and join operations correspond to the set theoretical intersection and union. The top element is the set Ω, and the bottom element is the empty set $\emptyset$. Lattice complementation operation is just the set-theoretic complementation: $\mathrm{A}' = \Omega \backslash \mathrm{A}$.

Finite distributive lattice can be viewed as a lattice of sets, and this is expressed by Birkhoff's representation theorem [15]. A special kind of distributive lattice is a *Boolean lattice*, where each element has a unique complement.

Since not every element of a distributive lattice has a complement, a weaker notion, called *pseudo-complement*, is defined. For a lattice element $a \in P$, its *pseudo-complement* is an element $a^* \in P$, such that $a^* \wedge a = 0$ and a^* is the largest such element to do so (i.e., meet a to 0). It turns out that every element of a finite distributive lattice has a unique pseudo-complement; though it may not be true that $a^* \vee a = 1$.

From the definitions is it intuitively clear that a topology can be thought of as a distributive lattice of sets. In fact, there is a direct correspondence between the operations of closure and interior of a set endowed with a topology and the operation of pseudo-complementation. In terms of event modeling, the pseudo-complement corresponds to intuitionistic negation, or "refutation" [7].

Lattice theory point of view is useful due to the recent interest in lattice belief functions. We will discuss next the theory of belief functions on different types of lattice, and relate them to topological modeling.

IV. BELIEF FUNCTIONS AND PROBABILITIES ON A LATTICE

Dempster-Shafer theory of belief functions [1] assumes a finite set Ω of possible answers to a question, referred to as the *frame of discernment*. The events correspond to subsets of Ω, with their uncertainty quantified by the *basic probability assignment, or bpa* (also referred to as *basic belief assignment, or bba*), which is a function $m: 2^{\Omega} \rightarrow [0,1]$ such that

$$m(\emptyset) = 0, \tag{1}$$

and

$$\sum_{A \subseteq \Omega} m(A) = 1. \tag{2}$$

The key idea here is that not all subsets of Ω encode inferred information; some are basic events and can be assigned basic (elementary) probability mass. Those subsets that we assign non-zero $m()$ are called *focal elements*. This structure is called a *body of evidence*. Based on our discussion of lattice theory, the body of evidence is defined on the powerset of Ω, or equivalently, on a Boolean lattice.

A belief function $Bel: 2^{\Omega} \rightarrow [0,1]$ is computed from the *bpa* as follows:

$$Bel(A) = \sum_{B \subseteq A} m(B), \tag{3}$$

where $A \subseteq \Omega$. The *bpa* can be recovered from the belief function using the following formula,

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \backslash B|} Bel(B). \tag{4}$$

Thus there is a one-to-one correspondence between m and Bel. The belief functions satisfy the following properties:

$$Bel(\emptyset) = 0, \qquad Bel(\Omega) = 1,$$

$$Bel\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{I \subset \{1,2..n\}, I \neq \emptyset} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right). \tag{5}$$

The last property shows that belief functions are *non-additive* even in the case of two disjoined subsets A and B: $Bel(A \cup B) \geq Bel(A) + Bel(B)$, in contrast to the additivity of probability functions.

If, and only if, the focal elements of *bpa* are 1-element subsets ("singletons") of Ω, does Bel become equivalent and reduces to a probability function. Thus, a classical probability function is a case of a belief function with *bpa* assigned to singleton elements of the Boolean lattice. In this case, the function Bel becomes additive.

Given two bodies of evidence with *bpa*'s m_1 and m_2 that need to be combined or "fused", the *conjunctive consensus* is given by the following formula:

$$m_{1\cap 2}(A) = \sum_{B\cap C=A} m_1(B)m_2(C). \quad (6)$$

The result of (6) needs to be normalized in order to satisfy (2). Dividing $m_{1\cap 2}$ by the total conflicting mass, K achieves normalization. K is defined in (7).

$$K = \sum_{B\cap C=\emptyset} m_1(B)m_2(C) \quad (7)$$

Formulas (8) and (9) correspond to the Dempster's rule of combination. The un-normalized version (6) of the Dempster's rule is used in Transferable Belief Models (TBM) [16]. Other combination rules have been proposed as well [2], [17]. The rule (6) can easily be generalized for more than two *bpa's*.

Since Bel is not additive in general, it is usually transformed into a probability function using the *pignistic* transformation [16]. Rational decisions are made based on *pignistic* probabilities, or betting changes, defined as follows,

$$Bet_m(a) = \sum_{\{A|A\subseteq\Omega, a\in A\}} \frac{m(A)}{|A|}, \forall a \in \Omega. \quad (8)$$

It turns out that the belief function framework can be generalized to the lattice setting [18]. For any poset $(P, \leq)$, and a function $f: P \to \mathbb{R}$, the Möbius transform [19] is the function $m: P \to \mathbb{R}$ such that

$$f(x) = \sum_{y\leq x} m(y). \quad (9)$$

This function can be computed as follows:

$$m(x) = \sum_{y\leq x} \mu(y,x) f(y), \quad (10)$$

using the Möbius function, defined recursively as follows:

$$\mu(x,y) = \begin{cases} 1 & if\ x = y \\ -\sum_{x\leq t<y} \mu(x,t) & if\ x < y \\ 0 & otherwise \end{cases}. \quad (11)$$

It is easy to show that the transforms (3) and (4) between the belief function and the basic probability assignment are special cases of the Möbius transform on a Boolean lattice. Given a lattice $(P, \leq)$, a function $Bel: P \to [0,1]$ is called a (generalized) belief function if $Bel(0) = 0$, $Bel(1) = 1$, Bel is strictly monotonic, and its Möbius transform is non-negative [20]. When a lattice admits a belief function, we can define a *bpa* on the lattice as well.

It has been shown that belief functions always exist on a lattice [18]. It has been further demonstrated in [8] and [21] that distributive lattices admit probability functions. In fact, in case when non-zero *bpa* is assigned only to the *join-irreducible* elements (elements that cannot be represented as joins of other lattice elements) of the distributive lattice, the resulting belief function is a probability function. It is the existence of probability functions on distributive lattices that enables our fusion scheme, as we will show in the next section.

V. Fusion Scheme on Topological Space

Consider a network of sensors used to perform automatic target detection and classification. The finite ground set Ω is the set of all possible targets. The space of possible events for Bayesian and DST frameworks is of course 2^Ω.

We assume that each of our sensors $S_s, s = 1..S$, can provide evidence supporting a certain subset, $X_s \subseteq \Omega$, and evidence supporting another subset $X_s^* \subseteq \Omega$, such that $X_s \cap X_s^* = \emptyset$. Such situation can occur when a sensor is trained to detect a certain feature of the target. The detection of this feature provides support for a certain subset of targets, X_s. The absence of this feature means that the target is not in X_s. However, the absence of the feature can only eliminate some of the remaining possibilities; in other words, it only supports a certain subset $X_s^* \subseteq \neg X_s$.

The set $\{X_s, X_s^*\}, s = 1..S$ is used as the set of generators for the topology $\mathcal{T}$. The topology is generated by finding all possible joins of all possible meets of the generating set. This is how our sensor network gives rise to a topological event space.

TABLE 1

Multi-sensor fusion in Topological Event Space
A. Given the set of possible events Ω, and the sensor network, generate open set topology $\mathcal{T}$
B. At each time step
1. Transform evidence coming from each sensor into *bpa* on $\mathcal{T}$
2. Fuse all *bpa's* from all sensors on $\mathcal{T}$, with conflict resolution
3. Transform the resulting *bpa's* into probability distribution by "Flow-Down" procedure
4. Compute probability for each singleton element of Ω
5. Make target class declaration decision based on MAP estimates of target class in Ω

The major steps in the fusion scheme are given in Table 1. The topology is generated off line, based on the knowledge of the sensors. At each time step, the evidence comes in from each sensor in the form of a posteriori probability distribution over the set of possible targets, which is our case is just two numbers: $\mathrm{P_s(X_s), P_s(X_s^*)}$. These numbers are transformed into the *bpa's*, m_s. The fusion of S *bpa's* is performed to result in the fused *bpa*, m. The resulting *bpa*, m, may have focal elements that are not join-irreducible, making the corresponding belief function non-additive. In order to ensure that the result of fusion is a probability function, we employ the "flow-down" procedure [11] that reassigns probability mass to only join-irreducible sets.

Once the "flow-down" is completed, the resulting belief function is used to make target class decision based on the maximum a posteriori probability. The detailed steps of *bpa* assignment procedure are given in Table 2. The details of our fusion algorithm are given in Table 3. The flow-down procedure is outlined in Table 4, and the decision procedure is given in Table 5. We will discuss these procedures next.

TABLE 2

Assignment of *bpa* in Topological Event Space
Given: $\mathcal{T}$ – open set topology generated by S-sensor network $P_s(X_s), P_s(X_s^*)$ – posterior probabilities of the declared target type X_s (and its refutation X_s^*) , for each sensor in the network, $s = 1..S$. **Initialize**: empty *bpa's* for $\mathcal{T}$: m_s **Returns**: m_s, $s = 1..S$
1. <u>For each sensor s</u> 2. CASE 1: $P_s(X_s) \geq P_s(X_s^*)$, evidence to the set with the feature 3. $m_s(X_s) = P_s(X_s)$ 4. $m_s(\Omega) = P_s(X_s^*)$ to ignorance 5. CASE 2: $P_s(X_s) > P_s(X_s^*)$, support the set without the feature 6. $m_s(X_s^*) = P_s(X_s^*)$ 7. $m_s(\Omega) = P_s(X_s)$ to ignorance

TABLE 3

Fusion of *bpa's* in Topological Event Space, with conflict redistribution
Given: $\mathcal{T}$ –topological event space generated by S-sensor network m_s- *bpa* obtained for each sensor in the network, $s = 1..S$. **Initialize**: empty *bpa's* for $\mathcal{T}$, m **Returns**: m
1. Form all possible S-intersections of the subsets of $\mathcal{T}$ with non-empty *bpa's*: $X = \bigcap_{s=1..S} X_s$, for all $X_s \in \mathcal{T}, m_s(X_s) > 0$ 2. <u>For each X</u> 3. compute conjunctive mass, $K_X = \prod_s m_s(X_s)$ 4. IF $X \neq \emptyset$ 5. THEN $m(X) = m(X) + K_X$ 6. ELSE (the mass K_X is a partial conflicting mass) 7. Proportionally distribute K_X to the non-empty sets in the current partial conflict $m(X_s) = m(X_s) + K_X \frac{m_s(X_s)}{\sum_{X_{s'} \neq \emptyset} m_{s'}(X_{s'})}$

The *bpa* assignment procedure in Table 2 makes a decision whether to support the set $\mathrm{X_s}$ or the set $\mathrm{X_s^*}$. The supporting mass is assigned to the corresponding set, and the remaining mass is assigned to Ω, which is the total ignorance.

The fusion procedure is the next major step. Table 3 outlines the procedure for a set of *bpa's* on $\mathcal{T}$.

Conflict resolution is a major difficulty to be overcome in any fusion procedure. Conflict occurs when two or more sensors assign supporting mass to subsets that are disjoint. The resulting mass is assigned to the empty set, which means that it supports an impossible event. Most fusion rules within DST and DSmT frameworks employ methodologies to redistribute the mass of conflict to other subsets. For example, the classical DST rule redistributes the total conflicting mass to all focal elements, by normalizing the mass assigned to the non-empty focal elements. Other fusion rules redistribute each conflicting mass separately.

Our procedure is modeled after the PCR6 [3], which show good performance in problems with high conflict [22]. The procedure starts by computing conjunctive consensus, defined in (12), which is a generalization of (6) for S sources.

$$m_\cap(X) = \sum_{X_1 \cap \ldots \cap X_S = X} m_1(X_1) \ldots m_S(X_S), \forall X \in \mathcal{T}. \quad (12)$$

This computation, defined for Boolean event space is still possible since $\mathcal{T}$ is closed under set intersections. The mass $m_\cap(X)$ is assigned to X. Some of the summands in (10) correspond to $X = \emptyset$. These are referred to as partial conflicts. The mass of each partial conflict is redistributed to the conflicting subsets as prescribed by PCR6 (step 7 in Table 3).

TABLE 4

Flow-Down Procedure
Given: $\mathcal{T}$ –topological event space generated by S-sensor network m – *bpa* on $\mathcal{T}$ **Initialize**: $X = \Omega$ **Returns**: m
1. Make a list of all children of X, $\mathcal{C}(X)$ 2. IF $\lvert\mathcal{C}(X)\rvert > 1$ (this is not a join-irreducible element) 3. THEN $m(y) = m(y) + \frac{m(X)}{\lvert\mathcal{C}(X)\rvert}, \forall y \in \mathcal{C}(X)$ 4. $m(X) = 0$ 5. Recur on each child element $y \in \mathcal{C}(X)$

Once the fused mass assignment is computed, the flow-down procedure is applied to redistribute the *bpa* to the join-irreducible elements of the topology. The join-irreducible elements are those that cannot be represented as joins of some other lattice elements. For example, in the lattice in Fig. 2, the join-irreducible elements are $\{c\}, \{d\}, \{a, c\}, \{b, c, d\}$. The procedure in Table 4 starts at the top lattice element and recurs on its child elements. The probability mass of any element that has more than one child is redistributed to its children. This procedure requires only one pass through the lattice, and it can be shown that the values of the obtained belief functions are located within the belief and plausibility intervals given by the original *bpa* [11].

The decision step requires computation of probability for each element of Ω. Denote by $\mathbb{P}$ the belief function computed from m using (7). Since we are operating on the lattice $\mathcal{T}$, this function is only defined on the singletons that belong to $\mathcal{T}$. In order to compute $\mathbb{P}$ for the singletons that are not included in $\mathcal{T}$, we take advantage of the fact that $\mathbb{P}$ is a probability function. For any $x \in \Omega, x \notin \mathcal{T}$, such that $\neg x \in \mathcal{T}$, we can write

$$\mathbb{P}(x) = 1 - \mathbb{P}(\neg x). \tag{13}$$

This way we can extend the probability function to the entire Ω and subsequently to the entire Boolean lattice 2^{Ω}. The decision is made based on the maximum of $\mathbb{P}$.

In the next section we will illustrate the presented fusion scheme with an example involving a sensor network with three sensors and four possible targets.

VI. Simulated Scenario for Sensor Fusion

Consider the following scenario. Three sensors are trained to detects certain target features, denoted by f_1, f_2, f_3. We assume that targets can belong to one of the four classes: a, b, c, d. We have a database containing previously observed targets, indicating target class and which target has which features, as shown in Table 5. Some of the values in this database are missing, as indicated by question marks.

Table 5

Object ID	f_1	f_2	f_3	Class
1	0	?	1	a
2	?	1	?	b
3	0	?	1	a
4	1	1	0	d
5	?	1	?	b
6	0	1	1	c
7	0	1	1	c
…	…	…	…	…

The missing values could mean that no decision was made on the feature being present or absent for a particular object. Suppose that grouping the data by class, we obtain the following table.

Table 6

Class	f_1	f_2	f_3
a	0	1	?
b	?	?	1
c	0	1	1
d	1	0	1

As we can see, for some classes, the values of certain features are missing. This means, for instance, that detecting feature f_1 can provide support for class *d*, and not detecting it can provide support for classes a and c, but will not provide any information about class *b*.

The classes supported by each sensor are summarized in Table 7.

Table 7

Sensor	Positive Class (X_s)	Negative Class (X_s^*)	Unknown Class
S_1	$\{d\}$	$\{a, c\}$	$\{b\}$
S_2	$\{a, c\}$	$\{d\}$	$\emptyset$
S_3	$\{b, c, d\}$	$\emptyset$	$\{a\}$

We will use the positive and negative sets in Table 7 to generate a topology. This is done by forming all possible intersections of the sets and then finding all possible unions. The resulting topology is shown in Fig. 1.

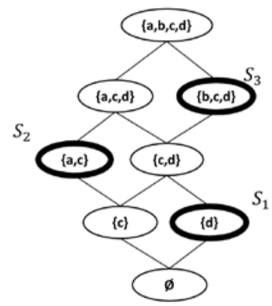

Figure 1 Topological event space corresponding to the sensor configuration in Table 7.

Each sensor serves as a binary classifier, declaring the presence or absence of the feature it's been trained on. Such classifier is characterized by a confusion matrix$C_f(D|T)$, given in (14). Here D stands for classifier declaration, and T stands for the true presence of the feature.

$$C_f \equiv \begin{pmatrix} P(D = f|T = f) & P(D = \neg f|T = f) \\ P(D = f|T = \neg f) & P(D = \neg f|T = \neg f) \end{pmatrix} \tag{14}$$

Each row in this matrix is a conditional probability distribution of the sensor declaration, conditioned on the presence and absence of the feature. We suppose that feature detector is designed and tested using Bayesian methods, which are optimal when there is no missing information. Using a priori distribution on the feature, $(P(f), P(\neg f))$, the following a posteriori probabilities of the feature are computed.

$$\begin{aligned} P_f(T = f|D = f) &= \frac{P(D = f|T = f)P(f)}{\sum_{T'} P(D = f|T' = f)P(f)} \\ P_f(T = \neg f|D = \neg f) &= \frac{P(D = \neg f|T = \neg f)P(\neg f)}{\sum_{T'} P(D = \neg f|T' = \neg f)P(\neg f)} \end{aligned} \tag{15}$$

The probabilities in (15) can be used as the source of the mass of evidence supporting certain subsets of our event space. For our example, we chose the following numbers for the mass of evidence supporting the positive and the negative classes.

Table 8

Sensor	$P(X_s)$	$P(X_s^*)$
S_1	0.9	0.6
S_2	0.9	0.6
S_3	0.9	0.6

Consider a case where the true target class is c. The sensors will provide the following support: S_1 to $\{a, c\}$, which is its negative clsss, S_2 to $\{a, c\}$ – its positive class, and S_3 to $\{b, c, d\}$. The *bpa's* and the results of computations are shown in Table 9.

flow-down procedure left the probability values within the original uncertainty interval.

TABLE 9 FUSION OF BPA'S EXAMPE, THE TRUE TARGET CLASS IS c (ZERO VALUES NOT SHOWN)

	$\emptyset$	a	b	c	d	ac	cd	acd	bcd	Ω
m_1					0.4					0.6
m_2						0.9				0.1
m_3									0.9	0.1
m				0.486	0.1092	0.2097			0.1865	0.0086
$\mathbb{BEL}$				0.486	0.1092	0.6957	0.5952	0.8049	0.7817	1
$\mathbb{PL}$				0.8908	0.3043	0.8908	1	1	1	1
m_f				0.4871	0.1103	0.2118		0	0.1908	
$\mathbb{P}$		0.2118	0.1908	**0.4871**	0.1103	0.6989	0.5973	0.8092	0.7882	1
$\mathbb{PL}_f$				0.8897	0.3011	0.8897	1	1	1	1
Bet		0.107	0.0643	0.6552	0.1735					
Bet_f		0.1059	0.0636	0.6566	0.1739					

The first three rows show the *bpa's* assigned by individual sensors. The next row shows the fused *bpa* m. We computed the belief and plausibility functions, $\mathbb{BEL}$ and $\mathbb{PL}$, in rows five and six. Row seven shows m_f, which is the *bpa* obtained after the flow-down procedure. Row eight and nine are the belief and plausibility functions corresponding to m_f. The last two rows show the pignistic probabilities (8) corresponding to m and m_f.

The probability $\mathbb{P}$ has been extended to sets a and b according to (13). As we can see, the maximum probability corresponds to class c, which is the correct answer.

VII. DISCUSSION AND CONCLUSION

This work considered topological event space modeling. The features of topological event spaces that are useful for probabilistic reasoning include being closed under unions and intersections, and the existence of probability functions, as shown in [20], [21]. The topological space can be significantly smaller than the corresponding Boolean space decreasing computational complexity. Finally, the set of topologies forms a lattice, which offers a potential principled way of fusing multiple event spaces, as proposed in [11].

TABLE 10 FUSION OF BPA'S EXAMPE, THE TRUE TARGET CLASS IS b (ZERO VALUES NOT SHOWN)

	$\emptyset$	a	b	c	d	ac	cd	acd	bcd	Ω
m_1				0	0.4	0	0	0	0	0.6
m_2				0	0	0.4	0	0	0	0.6
m_3				0	0	0	0	0	0.9	0.1
m				0.216	0.281	0.065	0	0	0.4002	0.0378
$\mathbb{BEL}$				0.216	0.281	0.281	0.497	0.562	0.8972	1
$\mathbb{PL}$				0.719	0.719	0.719	1	1	1	1
m_f				0.2207	0.2857	0.0744		0	0.4191	0
$\mathbb{P}$		0.0744	**0.4191**	0.2207	0.2857	0.2952	0.5064	0.5809	0.9256	1
$\mathbb{PL}_f$				0.7143	0.7048	0.7143	1	1	1	1
Bet		0.0419	0.1429	0.3914	0.4238					
Bet_f		0.0372	0.1397	0.3976	0.4254					

Table 10 shows the computations for true target class b. Individual sensors assign support as shown in the first three rows of the table. The rest of the numbers is computed in the same way as in Table 9. We see that the decision based on the maximum probability $\mathbb{P}$ is correct. Interestingly, the decision based on the maximum pignistic probability is incorrect in this case.

Observe that the values of $\mathbb{P}$ are always within the interval of the corresponding values of $\mathbb{BEL}$ and $\mathbb{PL}$, meaning that the

This approach was originally applied to cognitive modeling, in particular to model the human judgment of probability, as described in [9]. There, the open sets of topological space represent easily identifiable, or clear, instances of possible objects. The borders represent those instances that are not clear and may be ambiguous.

Topological event space modeling replaces the traditional classical logic utilized in probabilistic reasoning with intuitionistic logic. The primary feature of such logic is the impossibility to rely on the law of excluded middle. Statements can be proved based only on supporting evidence.

We investigate the applicability of topological event spaces in general fusion scenarios. We conjecture that situations where topological modeling and intuitionistic logic are applicable may arise in practical scenarios, such as the example in Sec. VI. We proposed a procedure for basic probability mass assignment and subsequent fusion. Preliminary results presented in this paper demonstrate that the proposed procedure results in probability function representing the fused bodies of evidence, and leads to correct decision making.

Although the initial results are encouraging, there are many unanswered questions that will be addressed in the future research. Here are the most immediate directions.

1. The current *bpa* assignment procedure may not be optimal in the sense of providing the best possible decision given the mixture of known and missing information for the problem at hand. For example, we may be required to introduce more sophisticated assignment that accounts for the relative size of missing vs. known information.

2. The current “flow-down” procedure that transforms the fused *bpa* into a join-irreducible assignment may not be optimal either. We may have to introduce certain weighting factors for mass reassignment, depending, for example, on the structure of the topology.

3. The current conflict resolution approach is essentially the PCR6 rule adapted to topological event space. PCR6 demonstrated good performance in recent studies; however it may have to be further modified for topological spaces.

4. Further studies need to be performed with various topologies, larger number of sensors and possible targets.

5. The proposed methodology need to eventually be tested on real world data.

6. The longer term goal of this research is to move toward the hierarchical scheme where multiple topologies are fused using the lattice of topologies. This will allow endowment of sensor network with a measure of reliability in the form of belief functions defined on the lattice of topologies, as discussed in [11].

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