GENERALIZATIONS OF CONJUGATE CONNECTIONS

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Generalizations of conjugate connections are studied in this paper. It is known that generalized conjugate connections, and semi-conjugate connections are generalizations of conjugate connections. To clarify relations of these connections, the notion of dual semi-conjugate connections is introduced. Then their relations are elucidated. Local triviality of generalized conjugate connection is also studied.

Keywords: Conjugate connection; Generalized conjugate connection; Semi-conjugate connection; Information geometry; Affine differential geometry.

Introduction

Geometry of conjugate connections is a natural generalization of geometry of Levi-Civita connections from Riemannian manifolds theory. Since conjugate connections arise from affine differential geometry and from geometric theory of statistical inferences, many studies have been carried out in the recent 20 years [1–3].

In this paper, we study generalizations of conjugate connections. Some of such generalizations have been introduced independently. The generelized conjugate connection was introduced in Weyl geometry [4]. The semi-conjugate connection was introduced in affine differential geometry [7]. In order to consider the relations between these connections, we introduce the dual semi-conjugate connection. Then we shall discuss the properties and the relations between these connections.

In the later part of this paper, we concentrate to study generalized conjugate connections. It is known that the generalized conjugate connection is invariant under gauge transformations. Hence, under suitable conditions, the generalized conjugate connection reduces to the standard conjugate connection. This property is called local triviality. Therefore, we give sufficient conditions for the generalized conjugate connection to have a local triviality.

1. Generalizations of conjugate connections

We assume that all the objects are smooth throughout this paper. We may also assume that a manifold is simply connected since we discuss local geometric properties on a manifold.

Let (M, g) be a semi-Riemannian manifold, and ∇ an affine connection on M. We can define another affine connection ∇^* by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z). \tag{1}$$

We call ∇^* the *(standard) conjugate connection* or the *(standard) dual connection* of ∇ with respect to g. It is easy to check that $(\nabla^*)^* = \nabla$.

In this paper, we consider generalizations of these conjugate connections.

Let (M, g) be a semi-Riemannian manifold, ∇ an affine connection on M, and $\widetilde{C}$ a $(0,3)$-tensor field on M. We define another affine connection $\widetilde{\nabla}^*$ by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \widetilde{\nabla^*}_X Z) + \widetilde{C}(X,Y,Z). \tag{2}$$

If the tensor $\widetilde{C}$ vanishes identically, then $\widetilde{\nabla}^*$ is the standard conjugate connection of ∇. Since the metric tensor g is symmetric, using twice Equation (2), we obtain the following proposition.

Proposition 1.1. $g(\widetilde{(\widetilde{\nabla}^*)}^*_X Y - \nabla_X Y, Z) = \widetilde{C}(X,Y,Z) - \widetilde{C}(X,Z,Y).$

Next, we shall define generalizations of the aforementioned conjugate connections.

Definition 1.1. Let (M, g) be a semi-Riemannian manifold, ∇ an affine connection on M, and τ a 1-form on M.

(1) The *generalized conjugate connection* [5,6] $\overline{\nabla}^*$ of ∇ with respect to g by τ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla}^*_X Z) - \tau(X)g(Y,Z). \tag{3}$$

(2) The *semi-conjugate connection* [5,7] $\hat{\nabla}^*$ of ∇ with respect to g by τ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \hat{\nabla}^*_X Z) + \tau(Z)g(X,Y). \tag{4}$$

(3) The *dual semi-conjugate connection* $\check{\nabla}^*$ of ∇ with respect to g by τ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \check{\nabla}^*_X Z) - \tau(X)g(Y,Z) - \tau(Y)g(X,Z). \tag{5}$$

The generalized conjugate connection is introduced in Weyl geometry to characterize Weyl connections [4]. The semi-conjugate connection arises naturally in affine hypersurface theory [7]. The dual semi-conjugate connection is introduced in this paper. As we will see later in this section, the dual semi-conjugate connection has dual property of the semi-conjugate connection.

From Proposition 1.1, $\overline{(\bar{\nabla}^*)}^* = \nabla$ holds for a generalized conjugate connection. On the other hand, this equality does not hold for a semi-conjugate connection $\hat{\nabla}$, or a dual semi-conjugate connection $\check{\nabla}^*$.

To clarify relations among the connections $\overline{\nabla}^*, \hat{\nabla}^*$ and $\check{\nabla}^*$, let us recall the projective equivalence relation and the dual-projective equivalence relation of affine connections.

Suppose that ∇ and ∇' are affine connections on a semi-Riemannian manifold (M, g). We say that ∇ and ∇' are *projectively equivalent* if there exists a 1-form τ such that

$$\nabla'_X Y = \nabla_X Y + \tau(Y)X + \tau(X)Y. \tag{6}$$

We say that ∇ and ∇' are *dual-projectively equivalent* if there exists a 1-form τ such that

$$\nabla'_X Y = \nabla_X Y - g(X,Y)\tau^{\#}, \tag{7}$$

where $\tau^{\#}$ is the metrical dual vector field, i.e., $g(X, \tau^{\#}) = \tau(X)$.

The readers should not confuse that, even if ∇ and ∇' are projectively (or dual-projectively) equivalent, their dual connections ∇^* and $(\nabla')^*$ may not be dual-projectively (or projectively) equivalent, respectively.

Proposition 1.2. *Let (M, g) be a semi-Riemannian manifold, ∇ an affine connection on M, and ∇^* the standard conjugate connection of ∇ with respect to g. Suppose that an affine connection ∇' is projectively equivalent to ∇ by τ. Then the following relations hold.*

(1) *The generalized conjugate connection $\overline{\nabla'}^*$ of ∇' by τ is dual-projectively equivalent to ∇' by τ with respect to g.*
(2) *The dual semi-conjugate connection $\check{\nabla}'^*$ of ∇' by τ coincides with ∇^*, that is, $\check{\nabla}'^* = \nabla$.*
(3) *The semi-conjugate connection $\hat{\nabla}'^*$ of ∇' by τ is given by*

$$\hat{\nabla}'^*_X Y = \nabla^*_X Y + g(X, Y)\tau^{\#} + \tau(Y)X + \tau(X)Y.$$

Proof. Form Equations (1), (3) and (6), we obtain

$$\begin{aligned} Xg(Y, Z) &= g(\nabla'_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) - \tau(X)g(Y, Z) \\ &= g(\nabla_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) - \tau(Y)g(X, Z). \end{aligned}$$

The last equality implies that $\overline{\nabla'}^*$ is dual-projectively equivalent to ∇^* by τ with respect to g.

On the other hand, from Equations (1), (5) and (6), we obtain

$$\begin{aligned} Xg(Y, Z) &= g(\nabla'_X Y, Z) + g(Y, \check{\nabla}'^*_X Z) - \tau(Y)g(X, Z) - \tau(X)g(Y, Z) \\ &= g(\nabla_X Y, Z) + g(Y, \check{\nabla}'^*_X Z). \end{aligned}$$

This implies that the dual semi-conjugate connection $\check{\nabla}'^*$ by τ coincides with ∇^*.

From Equations (1), (4) and (6), we obtain the third statement. □

Following similar arguments as in Proposition 1.2, we obtain the following proposition.

Proposition 1.3. *Let (M, g) be a semi-Riemannian manifold, ∇ an affine connection on M, and ∇^* the standard conjugate connection of ∇ with respect to g. Suppose that an affine connection ∇' is dual-projectively equivalent to ∇ by τ. Then the following relations hold.*

(1) *The generalized conjugate connection $\overline{\nabla'}^*$ of ∇' by τ is projectively equivalent to ∇' by τ.*
(2) *The semi-conjugate connection $\hat{\nabla}'^*$ of ∇' by τ coincides with ∇^*, that is, $\hat{\nabla}'^* = \nabla$.*
(3) *The dual semi-conjugate connection $\check{\nabla}'^*$ of ∇' by τ is given by*

$$\check{\nabla}'^*_X Y = \nabla^*_X Y + g(X, Y)\tau^{\#} + \tau(Y)X + \tau(X)Y.$$

2. Generalized conjugate connections and gauge transformations

Here after, we concentrate on generalized conjugate connections.

Let (M, g) be a semi-Riemannian manifold, ∇ an affine connection on M, and ϕ a function on M. We consider a conformal change of the metric $\bar{g} := e^{\phi}g$. Denote by $\overline{\nabla}^*$ the standard conjugate connection of ∇ with respect to the conformal metric $\bar{g}$. Then we obtain

$$\begin{aligned} &X\bar{g}(Y,Z) = \bar{g}(\nabla_X Y, Z) + \bar{g}(Y, \overline{\nabla}^*_X Z) \\ &\iff Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla}^*_X Z) - d\phi(X)g(Y,Z). \end{aligned} \tag{8}$$

This implies that $\overline{\nabla}^*$ is the generalized conjugate connection of ∇ with respect to g by $d\phi$.

Let τ be a 1-form on M. Set

$$(\bar{g}, \bar{\tau}) := (e^{\phi}g, \tau - d\phi). \tag{9}$$

The pair $(\bar{g}, \bar{\tau})$ is called a *gauge transformation* of (g, τ).

Proposition 2.1. *The generalized conjugate connection is invariant under gauge transformations. That is, $\overline{\nabla}^*$ is the generalized conjugate connection of ∇ with respect to g by τ if and only if $\overline{\nabla}^*$ is the generalized conjugate connection of ∇ with respect to $\bar{g}$ by $\bar{\tau}$.*

Proof. From Equation (8) and (9), we easily obtain

$$\begin{aligned} &Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla}^*_X Z) - \tau(X)g(Y,Z) \\ &\iff X\bar{g}(Y,Z) = \bar{g}(\nabla_X Y, Z) + \bar{g}(Y, \overline{\nabla}^*_X Z) - \bar{\tau}(X)\bar{g}(Y,Z). \end{aligned}$$

□

We remark that the gauge invariance is an important notion in Weyl geometry. We can discuss Weyl geometry in terms of the generalized conjugate connections [4,8].

3. Local triviality of generalized conjugate connections and equiaffine structures

Suppose that $\overline{\nabla}^*$ is the generalized conjugate connection of an affine connection ∇ by a 1-form τ. Equation (8) implies that $\overline{\nabla}^*$ is reduced to the standard conjugate connection with respect to some conformal metric $\bar{g}$ if $\tau = d\phi$.

Definition 3.1. The generalized conjugate connection $\overline{\nabla}^*$ is called *locally trivial* if there exists a function ϕ such that $\overline{\nabla}^*$ is the standard conjugate connection with respect to some conformal metric $\bar{g} = e^{\phi}g$. That is, the equivalence (8) holds.

In order to elucidate the local triviality of generalized conjugate connections, we consider equiaffine structures on a manifold.

Definition 3.2. Let ∇ be a torsion-free affine connection on M. Let ω be a volume form on M, that is, ω is an n-form on M which does not vanish everywhere. We say that the pair (∇, ω) is an *equiaffine structure* on M if ω is parallel with respect to ∇, that is, $\nabla\omega = 0$. We say that the connection ∇ is *equiaffine*, and the volume form ω is *parallel* with respect to ∇.

Let R be the curvature tensor of ∇, and Ric the Ricci tensor of ∇, i.e., $\mathrm{Ric}(Y,Z) = \mathrm{tr}\{X \mapsto R(X,Y)Z\}$. In local coordinate expressions, the quantities are given by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial}{\partial x^k},$$
$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = \sum_{k=1}^{n} R_{lij}^{k} \frac{\partial}{\partial x^k}$$

with

$$R_{lij}^{k} = \frac{\partial \Gamma_{lj}^{k}}{\partial x^i} - \frac{\partial \Gamma_{li}^{k}}{\partial x^j} + \sum_{m=1}^{n} \left(\Gamma_{mi}^{k}\Gamma_{lj}^{m} - \Gamma_{li}^{m}\Gamma_{mj}^{k}\right). \tag{10}$$

Proposition 3.1. *Let ∇ be a torsion-free affine connection on M. Then the following conditions are equivalent.*

(1) ∇ is equiaffine.
(2) Ric is symmetric.
(3) $\displaystyle \frac{\partial}{\partial x^i}\left(\sum_{k=1}^{n} \Gamma_{jk}^{k}\right) = \frac{\partial}{\partial x^j}\left(\sum_{k=1}^{n} \Gamma_{ik}^{k}\right).$

Proof. The statement (1) $\iff$ (2) is Proposition 3.1 in Section 1 by Nomizu and Sasaki [1]. Contracting (10) to get the Ricci tensor, we can easily obtain (2) $\iff$ (3). □

For further information about equiaffine structures, see Zhang's paper [9], or Zhang and Matsuzoe's paper [10].

Proposition 3.2. *Let ∇ be a torsion-free affine connection on M, and $dv = \sqrt{g}dx^1 \wedge \cdots \wedge dx^n$ the standard volume element with respect to g.*

Then the connection is equiaffine if and only if there exists a function f such that

$$\frac{\partial}{\partial x^k}\left(\log\left(f\sqrt{g}\right)\right) = \sum_{j=1}^{n} \Gamma_{kj}^{j}. \tag{11}$$

Proof. Recall that $\sqrt{g}$ is a non-zero function on M. If ∇ is equiaffine, then the condition (3) in Proposition 3.1 holds. This is the integrability condition of f since $\sqrt{g}$ is a non-zero function.

On the other hand, if the formula (11) holds, then the condition (3) in Proposition 3.1 holds. This implies the desired result, and we may say that the function f is just the proportionality function between the parallel form ω and the volume form dv. □

In the following it makes sense to assume τ exact. If M is connected and simply connected, by Poincaré's lemma, there is a function ϕ on M such that $\tau = d\phi$. Since we are considering local properties on a manifold, we have the following result.

Theorem 3.1. *Let (M, g) be a connected and simply connected n-dimensional semi-Riemannian manifold. Let ∇ be a torsion-free affine connection on M, and τ a 1-form on M. Suppose that the generalized conjugate connection $\overline{\nabla}^*$ of ∇ is torsion-free. Consider the following three conditions:*

(1) τ is an exact 1-form.
(2) ∇ is equiaffine.
(3) $\overline{\nabla}^*$ is equiaffine.

Then any two of the above conditions imply the third one.

Proof. In a local coordinate expression, the definition of $\overline{\nabla}^*$ in (3) is written

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki,j} + \overline{\Gamma}^*_{kj,i} - \tau_k g_{ij}.$$

Contracting the above equation with g^{ij}, we have

$$\sum_{i,j=1}^{n} g^{ij}\frac{\partial g_{ij}}{\partial x^k} = \sum_{i=1}^{n}\Gamma_{ki}^{i} + \sum_{i=1}^{n}\overline{\Gamma}^{*\,i}_{\ ki} - n\tau_k.$$

Let $\nabla^{(0)}$ be the Levi-Civita connection with respect to g. Using that

$$\sum_{i=1}^{n}\Gamma^{(0)\ i}_{ki} = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}\frac{\partial g_{ij}}{\partial x^k},$$

we obtain

$$n\tau_k = \sum_{j=1}^{n} \left\{ \Gamma_{kj}^{j} + \overline{\Gamma}_{kj}^{*\ j} - 2\Gamma_{kj}^{(0)\ j} \right\}.$$

Differentiating with respect to x^i and then x^k, subtracting, yields

$$n\left(\frac{\partial \tau_k}{\partial x^i} - \frac{\partial \tau_i}{\partial x^k}\right) = \sum_{j=1}^{n} \left\{ \left(\frac{\partial}{\partial x^i}\Gamma_{kj}^{j} - \frac{\partial}{\partial x^k}\Gamma_{ij}^{j}\right) + \left(\frac{\partial}{\partial x^i}\overline{\Gamma}^{*j}{}_{kj} - \frac{\partial}{\partial x^k}\overline{\Gamma}^{*j}{}_{ij}\right) \right\}$$
$$- 2\sum_{j=1}^{n} \left(\frac{\partial}{\partial x^i}\Gamma_{kj}^{(0)\ j} - \frac{\partial}{\partial x^k}\Gamma_{ij}^{(0)\ j}\right).$$

The last parenthesis vanishes since the Levi-Civita connection $\nabla^{(0)}$ is equiaffine (it admits a parallel volume element $dv = \sqrt{g}dx^1 \wedge \cdots \wedge dx^n$, or, equivalently, it has a symmetric Ricci tensor). Then

$$n\left(\frac{\partial \tau_k}{\partial x^i} - \frac{\partial \tau_i}{\partial x^k}\right) = \sum_{j=1}^{n} \left\{ \left(\frac{\partial}{\partial x^i}\Gamma_{kj}^{j} - \frac{\partial}{\partial x^k}\Gamma_{ij}^{j}\right) + \left(\frac{\partial}{\partial x^i}\overline{\Gamma}^{*j}{}_{kj} - \frac{\partial}{\partial x^k}\overline{\Gamma}^{*j}{}_{ij}\right) \right\}.$$

If any two of the above parentheses vanish, then the third one must vanish. Applying Proposition 3.1 leads to the desired result. □

From Theorem 3.1 and Proposition 3.1, we obtain the following.

Corollary 3.1. *Suppose that ∇ and $\overline{\nabla}^*$ are torsion-free. Consider the following three conditions:*

(1) τ is an exact 1-form.
(2) ∇ has a symmetric Ricci tensor.
(3) $\overline{\nabla}^*$ has a symmetric Ricci tensor.

Then any two of the above conditions imply the third one.

Let $\omega = f dv$, and $\overline{\omega}^* = \overline{f}^* dv$ be the volume elements associated with the generalized conjugate connections ∇ and $\overline{\nabla}^*$, with $f, \overline{f}^*$ functions, *i.e.*,

$$\nabla\omega = 0, \qquad \overline{\nabla}^*\overline{\omega}^* = 0,$$

where $dv = \sqrt{g}\, dx$ denotes the Riemannian volume element. We shall consider the following notations

$$[\omega] := f, \qquad [\overline{\omega}^*] := \bar{f}^*.$$

Theorem 3.2. *Let $\tau = d\phi$. Then*

$$[\omega][\overline{\omega}^*] = Ce^{n\varphi}, \tag{12}$$

with $C > 0$, constant.

Proof. Applying Proposition 3.2 to formula (11) yields

$$n\tau_k = \frac{\partial}{\partial x^k}(\log(f\sqrt{g})) + \frac{\partial}{\partial x^k}\left(\log(\bar{f}^*\sqrt{g})\right) - 2\frac{\partial}{\partial x^k}(\log\sqrt{g}).$$

Using $\tau_k = \frac{\partial \phi}{\partial x^k}$, the above relation can be written as an exact equation

$$\begin{aligned} \frac{\partial}{\partial x^k}\left(n\phi - \log(f\sqrt{g}) - \log(\bar{f}^*\sqrt{g}) + 2\log\sqrt{g}\right) = 0 \Longleftrightarrow \\ \frac{\partial}{\partial x^k}\left(n\phi - \log(f\bar{f}^*)\right) = 0 \Longleftrightarrow \\ n\phi - \log(f\bar{f}^*) = c \quad \text{constant} \Longleftrightarrow \\ f\bar{f}^* = Ce^{n\phi}, \end{aligned}$$

with $C = e^{-c}$. □

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